sequence

1. $a_n \le b_n \le c_n \quad \forall n \ge N \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \Rightarrow \lim_{n\to\infty} b_n = L$

[e.g.] $\lim_{n\to\infty} \frac{n!}{n^n}$

$$\frac{1}{n^n} \le \frac{n!}{n^n} \le \frac{n^{n-1}\cdot 1}{n^n} = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{n^n} = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

2. ☆ $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. if $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow a_n \to 0$ $\quad a_n \to 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ converges ($\frac{1}{n}$)

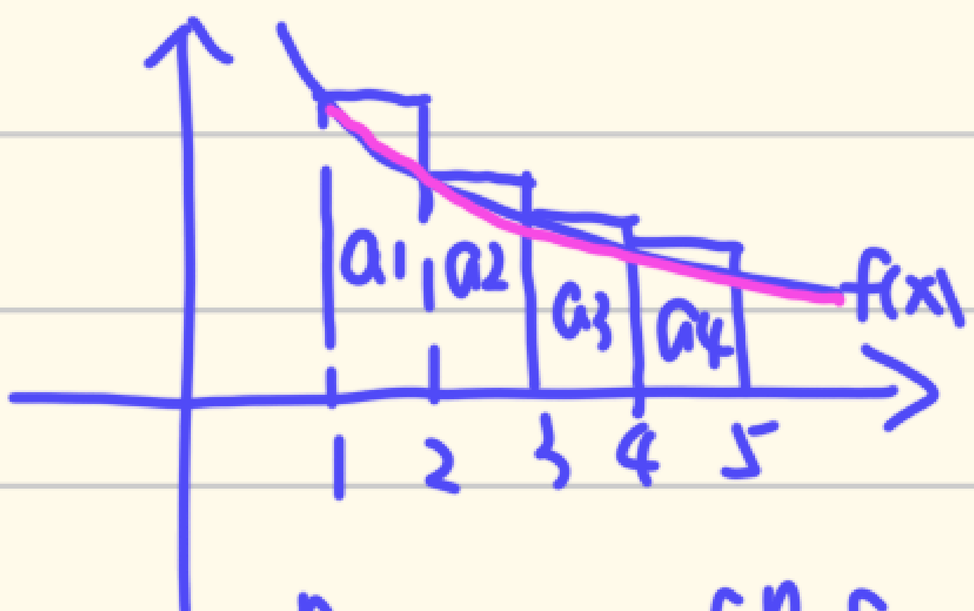
nth-term test $\quad$ if $\lim_{n\to\infty} a_n$ DNE $/ \ne 0 \Rightarrow$ diverges

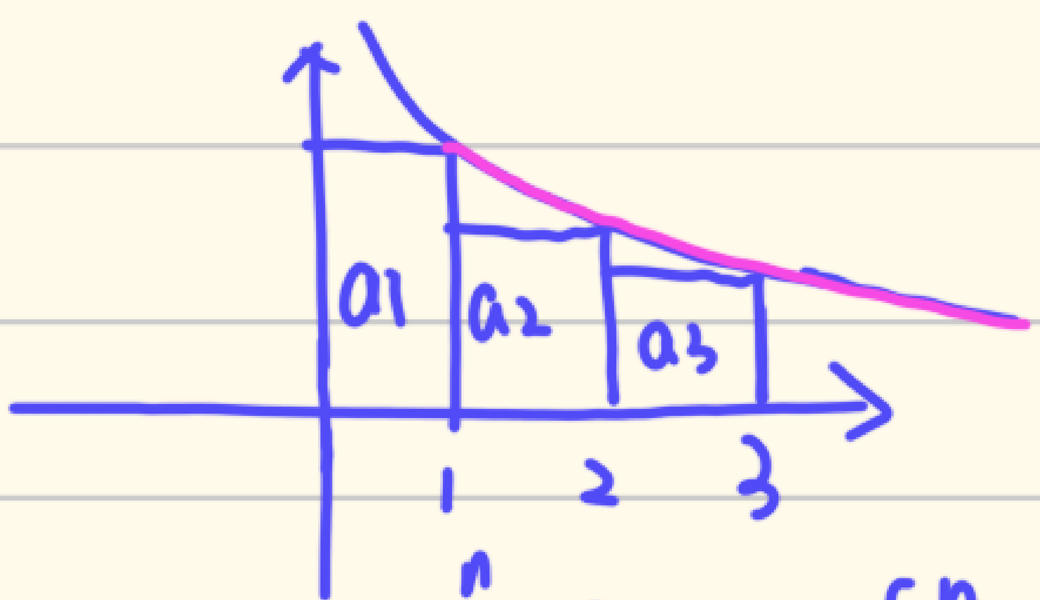
4. integral test

- $\sum_{n=1}^{\infty} a_n$ (非负) converges $\Leftrightarrow$ 部分项和有上界

[e.g.] $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \cdots \ge 1 + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$

- $a_n = f(n)$ $\quad f(n)$非负,递减 $\quad \sum_{n=N}^{\infty} a_n \longleftrightarrow \int_N^{\infty} f(x)\,dx$ both conv. or div.





① $\sum_{i=1}^{n} a_i > \int_1^n f(x)\,dx$ $\quad$ ② $\sum_{i=2}^{n} a_i < \int_1^n f(x)\,dx$ $\quad$ $S_n = \sum_{i=1}^{n} a_i$

$$\Rightarrow \int_1^n f(x)\,dx < S_n < \int_1^n f(x)\,dx + a_1$$

- $R_n = S_\infty - S_n = \sum_{i=n+1}^{\infty} a_i \Rightarrow \int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n+1}^{\infty} f(x)\,dx + a_{n+1} < \int_n^{\infty} f(x)\,dx$

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \le S = R_n + S_n \le S_n + \int_n^{\infty} f(x)\,dx$$

$$S \approx S_n + \frac{1}{2}\left(\int_n^{\infty} f(x)\,dx - \int_{n+1}^{\infty} f(x)\,dx\right) \le \frac{1}{2} a_n$$

5. comparison test

① $d_n \le a_n \le c_n$ (非负)

$\sum c_n$ conv. $\Rightarrow \sum a_n$ conv.

$\sum d_n$ div. $\Rightarrow \sum a_n$ div.

② $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ $\quad$ both conv./div. $\quad$ $\exists c > 0 \; a_n < c b_n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, $\sum b_n$ conv. $\Rightarrow \sum a_n$ conv.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ $\quad \sum b_n$ div. $\Rightarrow \sum a_n$ div.

5. $\sum |a_n|$ converges $\longrightarrow \sum a_n$ converges [逆命题不成立]

↳ e.g. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

$a_n + |a_n| = \begin{cases} 0, & a_n < 0 \\ 2|a_n|, & a_n \geq 0 \end{cases} \Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|$

$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ converges

6. ratio test

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$

① $\rho < 1$ abs. conv.

② $\rho > 1$ div

③ $\rho = 1$ inconclusive

$\exists N > n$.

$\Rightarrow \sum a_n = \sum_{i=0}^{\infty} \rho^k |a_N|$

$\Rightarrow \sum_{n=N} |a_n|$ conv. $\Rightarrow \sum_{n=1}^{\infty} |a_n|$ conv.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ conv.

7. root test

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$

① $\rho < 1$ abs. conv. $\Rightarrow |a_n| < r^n$

② $\rho > 1$ div

③ $\rho = 1$ inconclusive

8. alternating series test

$\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ conv. if ① $u_n \geq u_{n+1} > 0$ for $n \geq N$ ② $u_n \to 0$

$S_{2m} = u_1 - u_2 + u_3 - u_4 + \cdots$

$= u_1 - (u_2 - u_3) - (u_4 - u_5) - u_{2m} \leq u_1$

$\Rightarrow 0 \leq S_{2m} \leq u_1$ $\quad S_{2m+2} = S_{2m} + u_{2m+1} - u_{2m+2} \geq S_{2m}$

S_{2m} is nodecreasing and bounded from above $\Rightarrow \lim_{m\to\infty} S_{2m} = L$

$\lim_{m\to\infty} S_{2m+1} = \lim_{m\to\infty} (S_{2m} + \overset{\to 0}{u_{2m+1}}) = L$ $\Rightarrow \lim_{n\to\infty} S_n = L$

error: $|L - S_n| < |u_{n+1}|$ $\quad L - S_n$ 符号与 $(-1)^{n+2} u_{n+1}$ 符号相同

9. $\boxed{\text{abs.}}$ conv. $\{a_n\}$ 重排后为 $\{b_n\}$ 仍有 abs. conv.

10. power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ (a power series about $x = a$)

☆ $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ $(-1 < x < 1)$

if $\sum_{n=0}^{\infty} c_n (x-a)^n$ $\begin{cases} \text{conv. at } x = x_0 \Rightarrow \text{abs. conv. with } |x-a| < |x_0 - a| \\ \text{div. at } x = x_0 \Rightarrow \text{div. } |x-a| > |x - x_0| \end{cases}$

收敛半径 $R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$ (用ratio / root test 均可)

① $0<R<\infty$

$|x-a|<R \Rightarrow \limsup_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} = \frac{|x-a|}{R} < 1$ abs. conv.

$|x-a|>R \Rightarrow \frac{|x-a|}{R} > 1$ div.

$|x-a|=R \Rightarrow$ need to be tested extra. ☆ Check is important

② $R=\infty$ $\frac{|x-a|}{R}=0$ abs. conv.

③ $R=0$ $x=a$ conv. $x\neq a$ $\frac{|x-a|}{R}=\infty>1$ div.

· 级数乘法定理 $A(x)=\sum_{n=0}^{\infty} a_n x^n \ (|x|<R_1)$ $B(x)=\sum_{n=0}^{\infty} b_n x^n \ (|x|<R_2)$

$\Rightarrow C_n=\sum_{k=0}^{n} a_k b_{n-k}$ $C(x)=\sum_{n=0}^{\infty} C_n x^n$ abs. conv. $(|x|<R)$

$R\geq \min\{R_1, R_2\}$

类似卷积 convolution

· $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ $(a-R<x<a+R)$

$f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^n$ $f''(x)$ .... converges $(a-R<x<a+R)$

· $\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$ 例外 $\longrightarrow$ 求导后不满足a sum of positive integer power of x

· $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ $(a-R<x<a+R)$

$\int f(x)\,dx=\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}+C$ converges $(a-R<x<a+R)$

[e.g.1] $f(x)=\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x-\frac{x^3}{3}+\frac{x^5}{5}\cdots (-1\leq x\leq 1)$ [Identify the function]

$f'(x)=1-x^2+x^4+\cdots=\frac{1}{x^2+1}$ $(-1<x<1)$

$\int f'(x)\,dx=\tan^{-1}x+C$ $f(0)=0\Rightarrow C=0$

$\Rightarrow f(x)=\tan^{-1}x$ $(-1<x<1)$

[e.g.2] $f(x)=\sum_{n=0}^{\infty} (-1)^n x^n=\frac{1}{1+x}$ $(|x|<1)$

$\int \frac{1}{1+x}\,dx=\sum_{n=0}^{\infty} \frac{1}{n+1}(-1)^n x^{n+1}+C$ $(|x|<1)$

$=\ln|x+1|$

$x=0 \Rightarrow \ln|x+1|=0 \Rightarrow C=0$

$\Rightarrow \ln t=\sum_{n=0}^{\infty} \frac{1}{n+1}(-1)^n(t-1)^{n+1}$ $(0<t<2)$

• taylor series $(x=a)$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

if $x=0 \Rightarrow$ called maclaurin series (麦克劳林)

taylor polynomial of order n $\quad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

taylor's Formula $(x=a处)$

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x) \qquad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

ξ is between a and x

$R_n(x) \to 0 \Rightarrow$ conv.

[e.g.] $f(x)=e^x$ $(x=0)$ conv. for every real value of x

$$R_n(x) = \frac{e^{\xi}}{(n+1)!}x^{n+1}$$ ξ is between 0 and x

① $x<c<0$ $\quad 0 \le |R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \quad \lim_{n\to\infty} |R_n(x)| = 0$

② $0<c<x$ $\quad 0 \le |R_n(x)| \le e^x \left|\frac{x^{n+1}}{(n+1)!}\right| = 0 \quad \lim_{n\to\infty} |R_n(x)| = 0$

$\hookrightarrow e^c < e^x$. e^x is a constant

• $\exists M>0$ $\quad |f^{(n+1)}(\xi)| \le M$ $\quad \xi$ is between a and x

$$0 \le |R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!} = 0 \quad \Rightarrow \text{conv.}$$

• $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{i\theta} = \sum_{i=0}^{\infty} \frac{(i\theta)^n}{n!} = \left(1 + \frac{i^2\theta^2}{2} + \frac{i^4\theta^4}{4!} + \cdots\right) + \left(\frac{i\theta}{1} + \frac{i^3\theta^3}{3!} + \frac{i^5\theta^5}{5!} + \cdots\right)$$

$$= 1 - \frac{1}{2}\theta^2 + \frac{\theta^4}{4!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

• $e^{i\pi} + 1 = \cos\pi + i\cdot\sin\pi + 1 = -1 + 0 + 1 = 0$

talor <=> power series

[e.g.] 将 $f(x) = \frac{x-1}{4-x}$ 展开为 $(x-1)$的幂级数, 并求 $f^{(n)}(1)$.

$$f(x) = (x-1)\cdot\frac{1}{3-(x-1)} = (x-1)\cdot\frac{1}{3}\cdot\frac{1}{1-\frac{x-1}{3}} = (x-1)\cdot\sum_{i=0}^{\infty}\frac{1}{3}\left[1+\frac{x-1}{3}+\cdots+\left(\frac{x-1}{3}\right)^n\right]$$

$$= \frac{1}{3}\left[(x-1) + \frac{(x-1)^2}{3} + \cdots + \frac{(x-1)^{n+1}}{3^n}\right] \quad |x-1|<3$$

$$f(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$a_n = \frac{f^{(n)}(1)}{n!} \quad \Rightarrow f^{(n)}(1) = \frac{n!}{3^n}$$

专题一 级数 [常数项级数 → 函数项级数]

1. $\{a_n\}$ bounded and monotonic $\Rightarrow$ $\{a_n\}$ converges

2. $\{a_n\}$ 部分和 partial sum $\{S_n\}$ [$a_n > 0$]

$\sum_{i=1}^{\infty} a_n$ converges $\Leftrightarrow$ $\{S_n\}$ bounded

① $\sum_{i=1}^{\infty} a_n$ converges $\rightarrow$ $\lim_{n\to\infty} S_n$ exists $\rightarrow$ $\{S_n\}$ bdd.

② $\{S_n\}$ bdd. $a_i \geq 0$. $\{S_n\}$ increasing, $\{S_n\}$ bdd.

$\Rightarrow$ 单调有界. $\sum_{i=1}^{\infty} a_n$ converges

3. n-th term test

$\sum_{i=1}^{\infty} a_i$ converges $\Rightarrow$ $\lim_{n\to\infty} a_i = 0$

let $S = \sum_{i=1}^{\infty} a_i$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(a_n - a_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$

[$\lim_{n\to\infty} a_i = 0 \not\Rightarrow \sum_{i=1}^{\infty}$ converges e.g. $a_n = \frac{1}{n}$]

4. comparison test ($0 \leq a_i \leq b_i$)

$\sum_{i=1}^{\infty} b_i$ converges $\rightarrow \sum_{i=1}^{\infty} a_i$ converges

$\sum_{i=1}^{\infty} a_i$ diverges $\rightarrow \sum_{i=1}^{\infty} b_i$ diverges

① $\sum_{n=1}^{\infty} a_i$ conv. $\Rightarrow \sum_{n=1}^{\infty} a_i^2$ conv. ($a_i > 0$)

Proof: $\lim_{n\to\infty} a_n = 0$

$\exists M > 0$, $a_n < M$ ($a_n \geq 0$)

$\Rightarrow a_n^2 = a_n \cdot a_n < a_n \cdot M$ conv.

② $\sum_{n=1}^{\infty} a_i$, $\sum_{n=1}^{\infty} b_i$ conv. $\Rightarrow \sum_{n=1}^{\infty} a_i b_i$ conv.

• $\sum_{i=1}^{\infty} b_i \rightarrow 6$ $\exists M > 0$ $\sum_{i=1}^{n} b_i \leq M \Rightarrow S_n = \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \leq M$ by 2. $\Rightarrow \sum_{i=1}^{\infty} a_i$ converges

[e.g. $a_n = -1$ $b_n = \frac{1}{n^2}$ $a_n < b_n$, $\sum b_n$ converges, $\sum a_n$ diverges]

5. limit comparison test [$a_n > 0$, $b_n > 0$]

① $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$. $\sum a_n$, $\sum b_n$ both converge/diverge

$\exists N \in \mathbb{Z}^+ . \forall n > N .$

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2} \Rightarrow \frac{c}{2} b_n < a_n < \frac{3}{2} c\, b_n \Rightarrow \text{by } 4.$$

② $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ $\quad \sum b_n$ converges $\Rightarrow \sum a_n$ converges

$\exists N \in \mathbb{Z}^+ . \forall n > N .$ $\left|\frac{a_n}{b_n} - 0\right| < 1$ $\quad -b_n < a_n < b_n$

③ $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ $\quad \sum b_n$ diverges $\Rightarrow \sum a_n$ diverges

$\exists N \in \mathbb{Z}^+ . \forall n > N .$ $\frac{a_n}{b_n} > 1 \Rightarrow a_n > b_n$

6. geometric series

$$\sum_{n=1}^{\infty} a r^{n-1} = \frac{a(1-r^{n+1})}{1-r} = \frac{a}{1-r} \quad \text{if } |r| < 1$$

7. ratio test

$a_n > 0 .$ $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$ $\begin{cases} \rho < 1 . & \text{converges} \\ \rho > 1 , & \text{diverges} \\ \rho = 1 . & \text{fail to test} \end{cases}$ $\quad \left(\left|\frac{a_{n+1}}{a_n}\right| = \rho \quad a_n \in \mathbb{R} \right)$

when $\rho < 1$

$\rho + \epsilon = r < 1$ (let $\epsilon \to 0^+$)

$\exists N \in \mathbb{Z}^+ \ \forall n > N ,$ $\left|\frac{a_{n+1}}{a_n} - \rho\right| < \epsilon \Rightarrow a_{n+1} < r a_n$

$$\Rightarrow \underbrace{a_{N+m+1}}_{b_n} < r a_{N+m} < \underbrace{r^m a_{N+1}}_{C_n}$$

$\sum C_n$ geometric series with $r<1$ and $b_n < C_n \Rightarrow b_n, C_n$ converge

$\sum b_n$ 与 $\sum a_n$ 相差有限项 $\Rightarrow \sum a_n$ converges

8. root test

$a_n > 0$ $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \rho$ $\begin{cases} \rho < 1 . & \text{converges} \\ \rho > 1 , & \text{diverges} \\ \rho = 1 . & \text{fail to test} \end{cases}$ $\quad \left(\sqrt[n]{|a_n|} = \rho \quad a_n \in \mathbb{R} \right)$

$\lim\limits_{n\to\infty} \sqrt[n]{n!} = \lim\limits_{n\to\infty} \frac{n}{e} = \infty$

$e^n = \sum\limits_{i=0}^{n} \frac{n^i}{i!} \geq \frac{n^n}{n!}$

when $\rho < 1 .$

$\left|\sqrt[n]{a_n} - \rho\right| < \epsilon \Rightarrow \sqrt[n]{a_n} < \rho + \epsilon \Rightarrow a_n < (\rho + \epsilon)^n$ $\quad r = \rho + \epsilon < 1 \Rightarrow$ converges

9. alternating series

$\sum_{i=1}^{\infty} (-1)^{n-1} a_i$ if $\begin{cases} a_n \geq a_{n+1} \\ \lim_{n\to\infty} a_n = 0 \end{cases} \Rightarrow s = \sum_{i=1}^{\infty} (-1)^{n-1} a_i$ converges

and $s \leq a_1$, $|r_n| \leq a_{n+1}$

$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n})$ increasing

$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \Rightarrow S_{2n} \leq a_1$

by 1. $\lim_{n\to\infty} S_{2n} = S \leq a_1$

$\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + a_{2n+1}) = S \leq a_1$

$r_n = S - S_n = \pm(a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} \cdots) \Rightarrow |r_n| \leq a_{n+1}$

10. $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

$-|a_n| \leq a_n \leq |a_n| \Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|$

by 4. $\Rightarrow \sum (a_n + |a_n|)$ conv.

$\sum a_n = \sum (a_n + |a_n| - |a_n|) = \sum (a_n + |a_n|) - \sum |a_n|$

$\Rightarrow \sum a_n$ converges

11. $\sum a_n$ abs. converges $\longrightarrow$ 重排后为 b_n $\sum b_n$ abs conv. and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$

$\sum a_n \begin{cases} \text{部分和 } S_n \\ \text{和 } S \end{cases}$ b_n 部分和 σ_n

$\forall n$, when $m \to \infty$

$S_m = a_1 + \cdots a_m$, 同时让 $b_1, \cdots b_n$ 均出现在S_m中. $\sigma_n \leq S_m \leq S$

$\begin{cases} \sum b_n = \lim_{n\to\infty} \sigma_n = \sigma \leq S \\ \sum a_n = \lim_{n\to\infty} S_n = S \leq \sigma \end{cases} \longrightarrow \sigma = S$ $(a_n, b_n > 0)$

for $a_n, b_n \in \mathbb{R}$

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$ $\quad \sum a_n + |a_n| = \sum b_n + |b_n|$ $\quad \sum |a_n| = \sum |b_n|$

$\Rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (b_n + |b_n|) - \sum_{n=1}^{\infty} |b_n| = \sum_{n=1}^{\infty} a_n$

12. power series

$\sum_{i=0}^{\infty} ax^i \quad |x|<1 \quad$ converges

<1> for $\sum_{n=0}^{\infty} a_n x^n$ $\begin{cases} x=x_0\ (x_0\neq 0) \text{ converges} \rightarrow |x|<|x_0|, \text{ abs. conv. } ① \\ x=x_0 \quad \text{div.} \rightarrow |x|>|x_0| \text{ div.} \end{cases}$

to prove ①:

$\lim_{n\to\infty} a_n x_0^n = 0 \qquad \exists M>0 \qquad |a_n x_0^n| \le M$

$\Rightarrow |a_n x^n| = \left|a_n x_0^n \cdot \frac{x^n}{x_0^n}\right| \le M \left|\frac{x}{x_0}\right|^n \Rightarrow$ when $|x|<|x_0|$, abs. conv

<2> $\sum_{n=0}^{\infty} a_n (x-a)^n$ $\begin{cases} |x-a|<R \text{ abs. conv.} \quad |x-a|>R \text{ div.} \\ \forall x \text{ . conv.} \Rightarrow R=\infty \\ \text{conv. only } x=a \Rightarrow R=0 \end{cases}$

<3> $\sum_{n=0}^{\infty} a_n x^n$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho \qquad R = \begin{cases} \frac{1}{\rho}, & \rho\neq 0 \\ \infty, & \rho=0 \\ 0, & \rho=\infty \end{cases}$

$\sum_{n=2}^{\infty} \frac{1}{n^\alpha (\ln n)^\beta}$

$\begin{cases} \alpha>1 & \text{conv.} \\ \alpha<1 & \text{div.} \\ \alpha=1 & \begin{cases} \beta>1 & \text{conv.} \\ \beta\le 1 & \text{div.} \end{cases} \end{cases}$

<4> term-by-term differentiation/integration thorem

$S(x) = \sum_{n=0}^{\infty} a_n x^n \quad (x\in I)$

$\begin{cases} \int_0^x S(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} & x\in I \\ S'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} & x\in I \end{cases}$

13. talor series

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n \qquad (x=0 \rightarrow$ maclaurin series$)$

$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-1)^{n+1} \qquad \lim_{n\to\infty} R_n(x) = 0$

$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = 0 \qquad R=\infty$

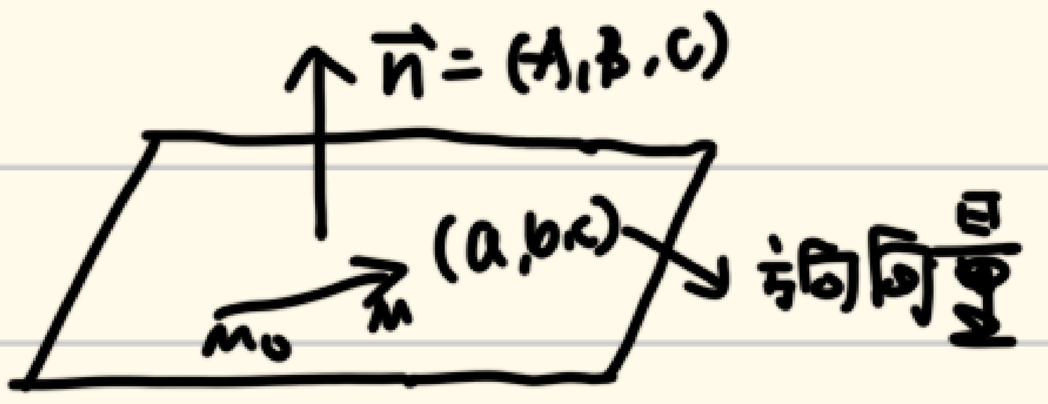
14. binomial series

$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k \qquad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$

$(-1<x<1)$

Space analytic geometry

1.



$\overrightarrow{M_0M}=(x-x_0,\ y-y_0,\ z-z_0)$

MM0:

$\overrightarrow{M_0M}\cdot\vec n = 0 \rightarrow A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

$(a,b,c)\parallel\overrightarrow{MM_0} \rightarrow$ MM0: $\frac{x-x_0}{a}+\frac{y-y_0}{b}+\frac{z-z_0}{c}=0$

2. $Proj_{\vec u}\vec v = \|v\|\cos\theta\cdot\frac{\vec u}{\|\vec u\|} = \frac{\vec u\cdot\vec v}{\|\vec u\|}\cdot\frac{\vec u}{\|u\|}$ $\vec v$在$\vec u$上的投影向量

3. cross product

① $\vec u\times\vec v=-\vec v\times\vec u$ $\quad\vec u\times(\vec v\times\vec w)=(\vec u\vec w)\vec v-(\vec u\vec v)\cdot\vec w$

Proof: $\vec v\times\vec w\perp\vec v/\vec w$ $\quad\vec u\times(\vec v\times\vec w)\perp\vec u/\vec v\times\vec w$

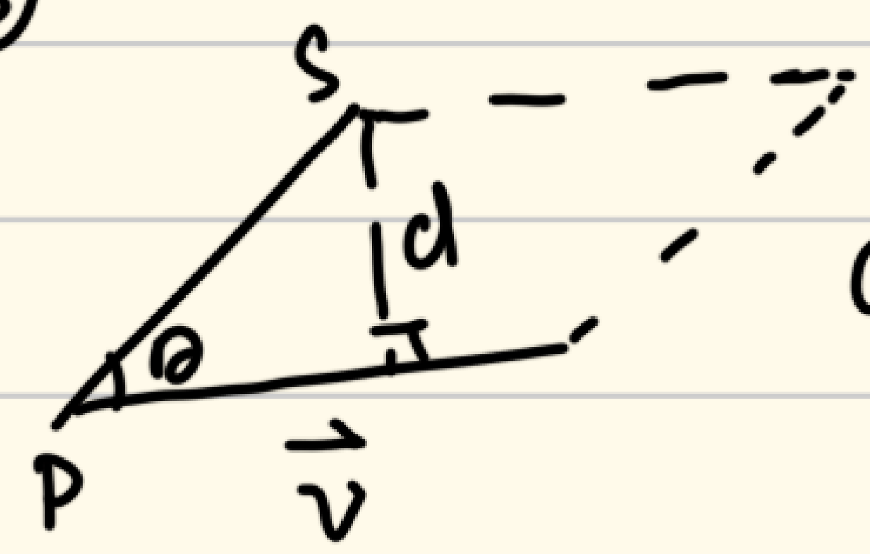
$\vec u\times\vec v\times\vec w=a\vec v+b\vec w$

$\vec u\cdot(\vec u\times\vec v\times\vec w)=\vec u(a\vec v+b\vec w)=0\Rightarrow\begin{cases}a=\vec u\vec w\\ b=-\vec u\vec v\end{cases}$

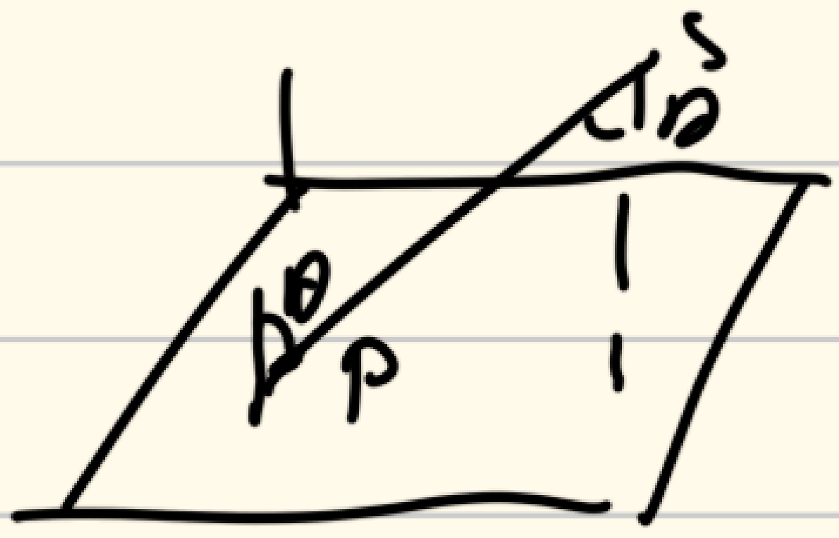
$\Rightarrow\vec u\times\vec v\times\vec w=(\vec u\vec w)\vec v-(\vec u\vec v)\vec w$

② $\vec u\times\vec v=(|\vec u||\vec v|\sin\theta)\vec n$

③

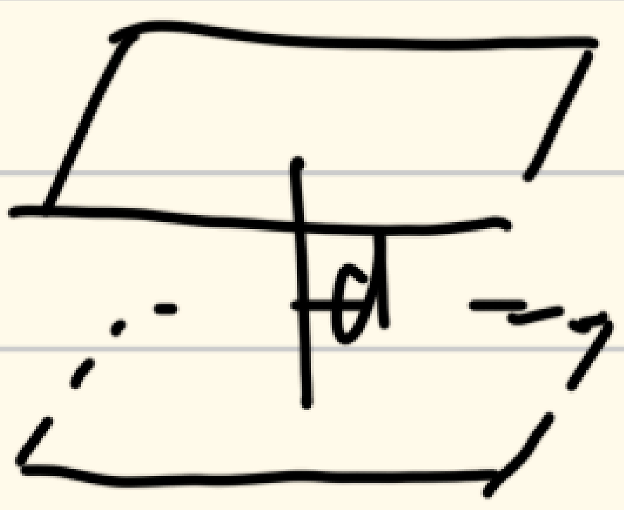


$d=\frac{\|\overrightarrow{PS}\times\vec v\|}{\|\vec v\|}$



$d=\frac{\|\overrightarrow{PS}\cdot\vec n\|}{\|\vec n\|}$

$Ax+By+Cz=D_k\ (k=1,2)$



$d=\frac{\|(A\cdot B\cdot C)(x_1-x_2,y_1-y_2,z_1-z_2)\|}{\sqrt{A^2+B^2+C^2}}=\frac{|D_1-D_2|}{\sqrt{A^2+B^2+C^2}}$

④ the intersection lines of two non-parallel planes

交线同时垂直于两平面法向量 $\quad\vec v=\vec{n_1}\times\vec{n_2}$

⑤ 平面束方程 $\quad L=\begin{cases}A_1x+B_1y+C_1z+D_1=0\\ A_2x+B_2y+C_2z+D_2=0\end{cases}$

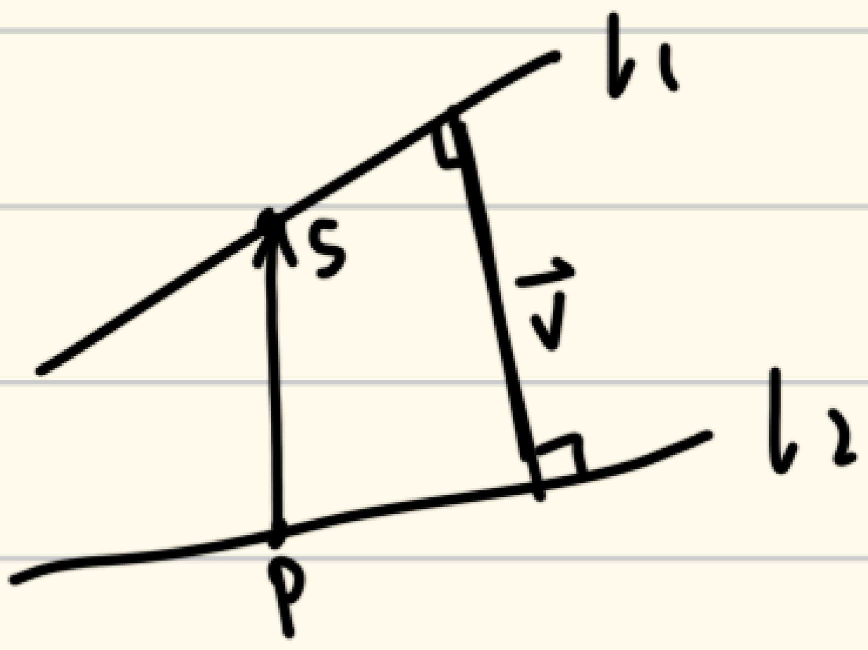
所有过L的平面：$A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0$

（不包括 $A_2x+B_2y+C_2z+D_2$ $\quad\vec n=\vec{n_1}+\lambda\vec{n_2}$ 进行线性组合）

⑥ distance between skew lines 异面直线距离

→ find the common perpendicular line 公垂线

$\vec{v} = \vec{v_1} \times \vec{v_2}$



$d = \dfrac{\|\vec{PS} \cdot \vec{v}\|}{\|\vec{v}\|}$

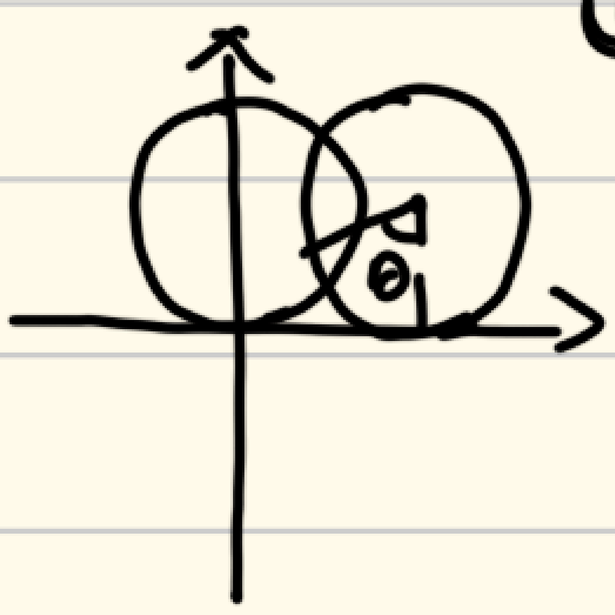
4. 参数式方程

$$\begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases}$$

$\dfrac{x - x_0}{m} = \dfrac{y - y_0}{n} = \dfrac{z - z_0}{p} = t$ 方向向量 $\vec{s} = (m, n, p)$

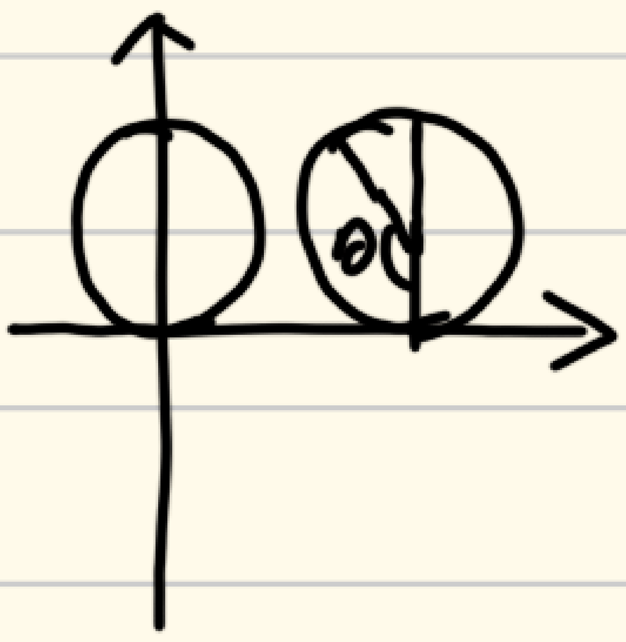
Parametrization $[r(t) \cdot r'(t) = 0 \Rightarrow r^2(t) = C \to (r^2(t))' = 0 \Rightarrow r(t) r'(t) = 0]$

1. cycloid 摆线



① $\theta \in (0, \frac{\pi}{2})$

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$$



② $\theta \in (\frac{\pi}{2}, \pi)$

$x = \theta r - r\cos(\theta - \frac{\pi}{2})$

$= r(\theta - \sin\theta)$

$y = r + r\sin(\theta - \frac{\pi}{2})$

$= r(1 - \cos\theta)$

$$\Rightarrow \begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$$

2. $\dfrac{d^2y}{dx}$ $\quad y' = \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\Rightarrow y'' = \dfrac{dy'}{dx} = \dfrac{dy'}{dt} \cdot \dfrac{1}{\frac{dx}{dt}} = \dfrac{\frac{d(y')}{dt}}{\frac{dx}{dt}}$

3.

$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx = \int_{t_0}^{t_1} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

与x轴围成的面积

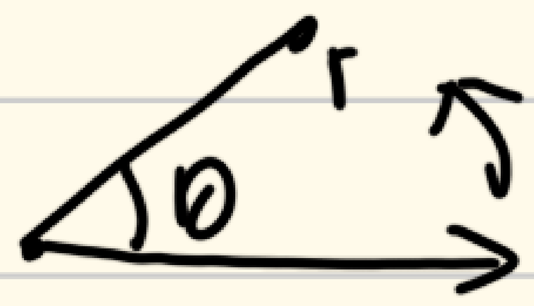
$A = \int_{x(t_0)}^{x(t_1)} y\, dx \Rightarrow dA = y\,dx = y(t) \cdot \dfrac{dx}{dt} \cdot dt = y(t) \cdot x'(t)\, dt$

$A = \int_{t_0}^{t_1} y(t)\, x'(t)\, dt$

绕x轴旋转的面积 $S = \int_a^b 2\pi y\, dL = \int_a^b \underbrace{2\pi y}_{\text{圆周长}} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

弧长微元 圆周长 实

4. polar coordinates



$$\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases} \Rightarrow \frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

arc length

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_a^b \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta$$

$$= \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Area of Fan-shaped Regions

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{2} \Delta\theta_i (f(\theta_i))^2 = \int_\alpha^\beta \frac{f^2(\theta)}{2}\, d\theta \qquad \left[\int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2)\, d\theta \right]$$

• smooth $\frac{d\vec{r}(t)}{dt}$ exists and is continuous and never $\vec{0}$ from $t=a$ to $t=b$.

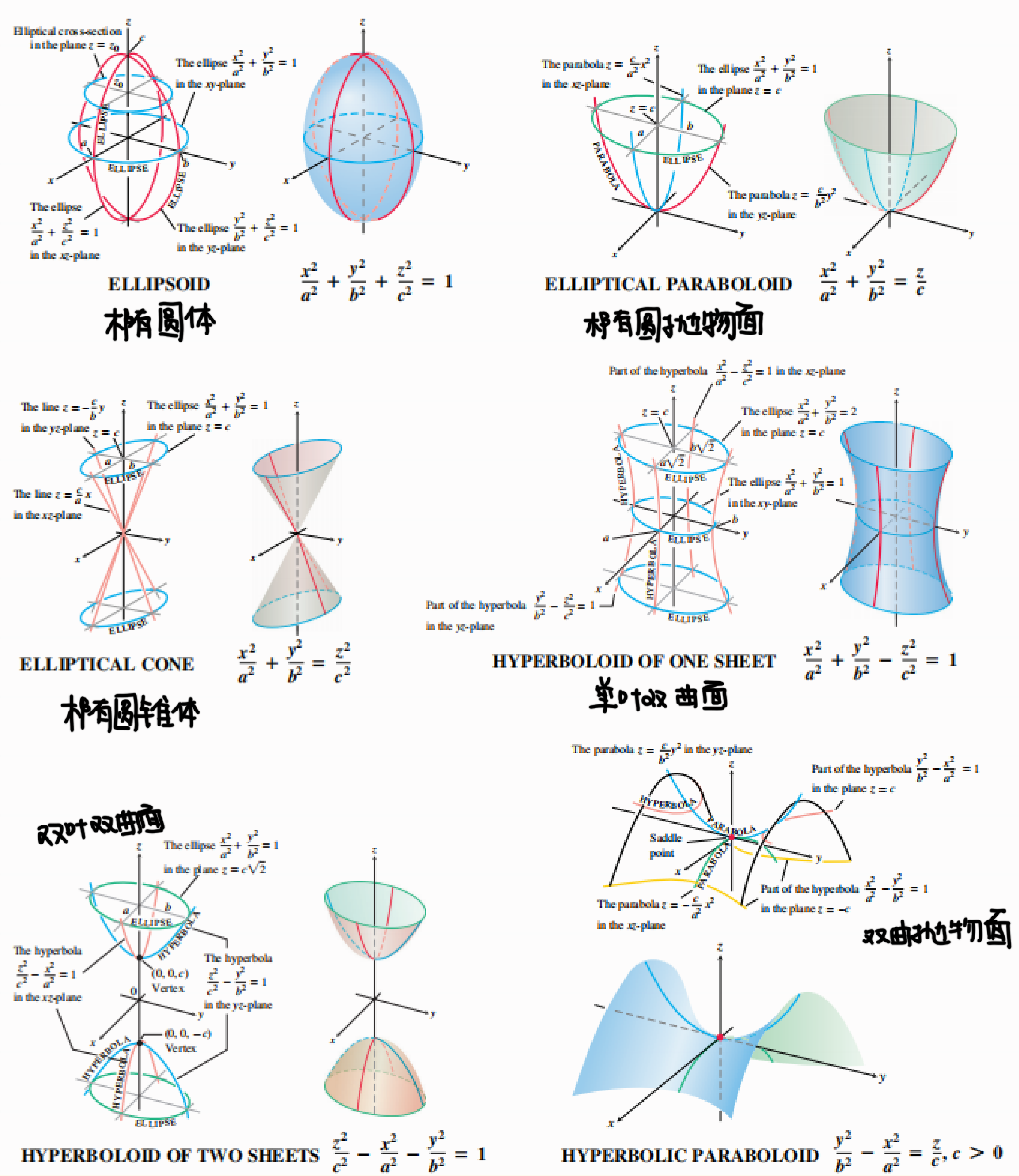
• curvature

$$k = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}\right| = \frac{1}{|r'(t)|}\left|\frac{dT}{dt}\right| \qquad \vec{N} = \frac{1}{k}\frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

$$\Rightarrow \vec{v} = \vec{i} + f'(x)\vec{j} \qquad |\vec{v}| = \sqrt{1+f'(x)^2} \qquad \vec{T} = \frac{1}{\sqrt{1+f'(x)^2}}\vec{i} + \frac{f'(x)}{\sqrt{1+f'(x)^2}}\vec{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \left|\frac{-f'(x)f''(x)}{(1+f'(x)^2)^{\frac{3}{2}}}\vec{i} + \frac{f''(x)}{(1+f'(x)^2)^{\frac{3}{2}}}\vec{j}\right| = \sqrt{\frac{f''(x)^2(f'(x)^2+1)}{(1+f'(x)^2)^3}} = \frac{|f''(x)|}{|f'(x)^2+1|} \qquad k = \frac{|f''(x)|}{[1+f'(x)^2]^{\frac{3}{2}}}$$

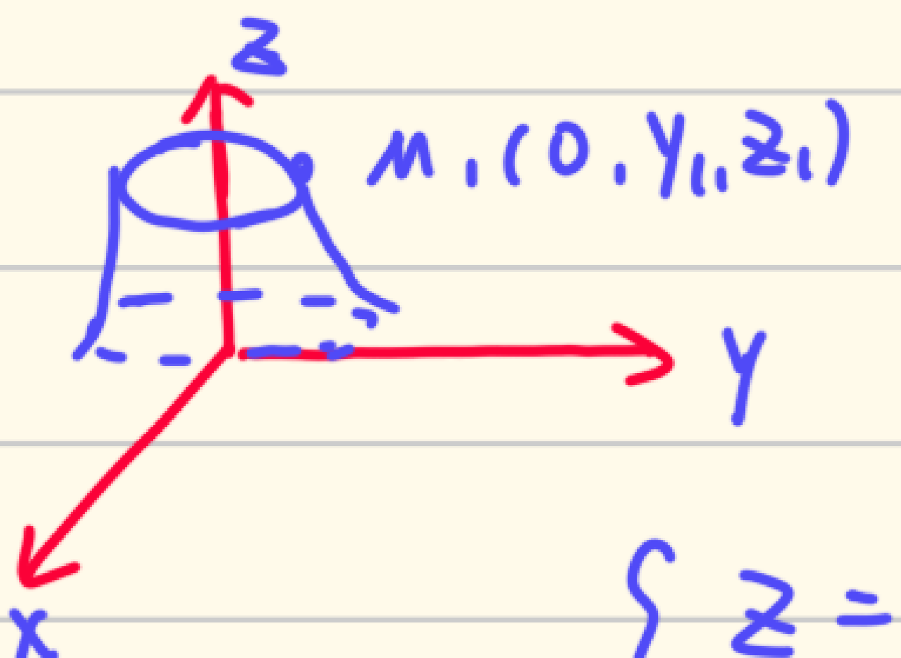
· quadratic surface



椭圆体　椭圆抛物面　椭圆锥体　单叶双曲面　双叶双曲面　双曲抛物面

· 曲线绕某一轴旋转的曲面方程　被　与缺失变量平方和后开根

$$\begin{cases} f(x,y)=0 \\ z=0 \end{cases}$$ 绕x轴　$f(x, \pm\sqrt{y^2+z^2})=0$



$L:\begin{cases} f(y,z)=0 \\ x=0 \end{cases}$ 绕z轴

$$\begin{cases} z=z_1 \\ x^2+y^2=y_1^2 \end{cases} \Rightarrow f(\pm\sqrt{x^2+y^2}, z)=0$$

- 证明二元函数 limit exists

1. ☆ sandwich theorm

[e.g.] $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} \Rightarrow |3y| \cdot \overbrace{\frac{x^2}{x^2+y^2}}^{\le 1} \le |3y| = 0$

2. ρ 极坐标

[e.g.] $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{r^2\cos\theta\sin\theta}{r^2} = \frac{1}{2}\sin 2\theta$ depend ons θ

DNE

- 判断可微

$$☆ \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{[f(x+\Delta x, y+\Delta y) - f(x,y)] - [f_x'(x,y)\Delta x + f_y'(x,y)\Delta y]}{\sqrt{\Delta x^2+\Delta y^2}} = 0$$

可微

[e.g.1] 判断 differentiable

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & x^2+y^2 > 0 \\ 0 & , x^2+y^2 = 0 \end{cases}$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(\Delta x+0, \Delta y+0) - f(x,y) - 0}{\sqrt{\Delta x^2+\Delta y^2}} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x \Delta y}{\Delta x^2+\Delta y^2}$$

let $\Delta y = k\Delta x \Rightarrow = \frac{k\Delta x^2}{(k^2+1)\Delta x^2} = \frac{k}{k^2+1}$ depends on k

$\to f(x,y)$ is not differentible on $(0,0)$ 可偏导 $\not\Rightarrow$ 可微

[e.g.2] 判断 ① differentiable ② $f_x'(x,y)$ continuous at $(0,0)$

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2} & (x,y)\ne(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

① $f'_x(0,0)=f'_y(0,0)=0$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(\Delta x+0,\Delta y+0)-f(0,0)-f'_x(0,0)\Delta x-f'_y(0,0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\sqrt{\Delta x^2+\Delta y^2}\,\overbrace{\sin\frac{1}{\Delta x^2+\Delta y^2}}^{有界}=0$$

$\Rightarrow$ differentiable at $(0,0)$

②
$$f'_x(x,y)=\begin{cases}2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} f'_x(x,y)=\underbrace{0-\text{DNE}}_{\text{DNE}}$$

$\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}$ $y=x\to\frac{1}{x}\cos\frac{1}{2x^2}$ 振荡 DNE

可微 $\not\to$ $f'_x(x,y)$, $f'_y(x,y)$ continuous

conclusion

☆ $f(x,y)$ at (x_0,y_0)

<1> $f'_x(x,y)$, $f'_y(x,y)$ is continuous
$\left[\lim_{\substack{x\to x_0\\ y\to y_0}} f'_x(x,y)=f'_x(x_0,y_0)\ /\ \lim_{\substack{x\to x_0\\ y\to y_0}} f'_y(x,y)=f'_y(x_0,y_0)\right]$

<1> $\xrightarrow{*}$ 可微; 可微 $\not\to$ <1>

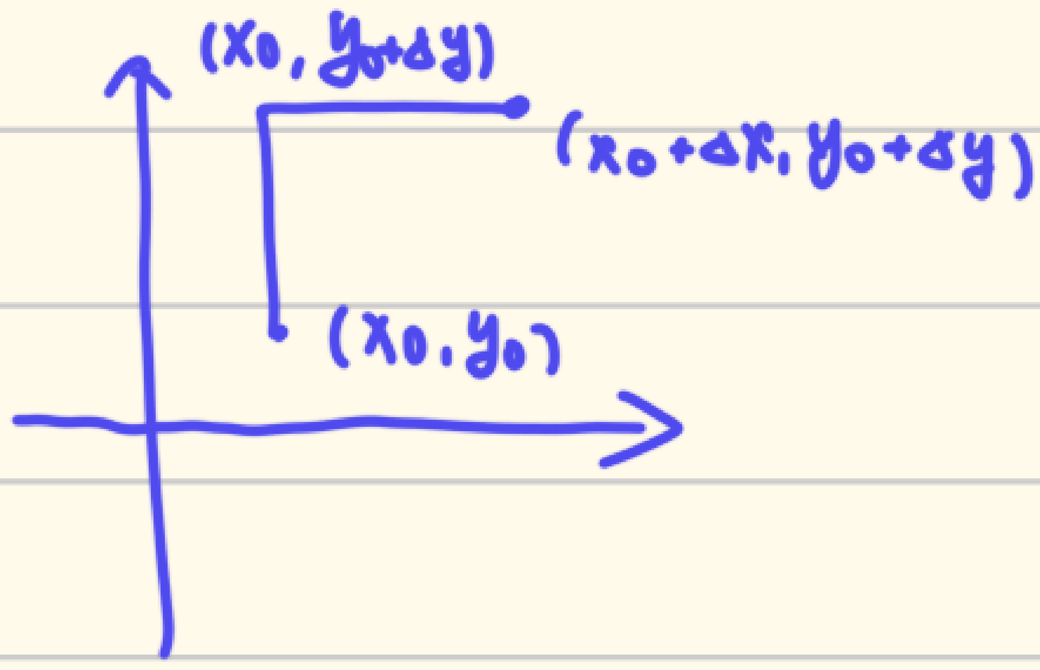
<2> 可微

可偏导 $\not\to$ 可微; 可微 $\to$ 可偏导

连续 $\not\to$ 可微; 可微 $\to$ 连续

<3> $f'_x(x_0,y_0)$, $f'_y(x_0,y_0)$ exists $\not\leftrightarrow$ 连续 $\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=f(x_0,y_0)$

* proof:



$$\begin{aligned}\Delta z&=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)\\&=\Delta z_x+\Delta z_y\\&=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0+\Delta y)\\&\quad+[f(x_0,y_0+\Delta y)-f(x_0,y_0)]\\&=\underline{f'_x(\xi_1,y_0+\Delta y)\Delta x+f'_y(x_0,\xi_2)\Delta y}\end{aligned}$$

连续 $\to$ 拉格朗日

ξ_1 is between x_0 and $x_0+\Delta x$, ξ_2 is between y_0 and $y_0+\Delta y_0$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f'_x(\xi_1,y_0+\Delta y)=f'(x_0,y_0)$$

$\Rightarrow f'_x(\xi_1,y_0+\Delta y)=f'_x(x_0,y_0)+\varepsilon_1$, $f'_y(x_0,\xi_2)=f'_y(x_0,y_0)+\varepsilon_2$ $(\varepsilon_1,\varepsilon_2\to 0)$

$$\Delta z=f'_x(x_0,y_0)\Delta x+f'_y(x_0,y_0)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$$
$$=f'_x(x_0,y_0)\Delta x+f'_y(x_0,y_0)\Delta y+o\left(\sqrt{\Delta x^2+\Delta y^2}\right)$$

$\Rightarrow z$ is differentiable at (x_0,y_0)

· Partial derivatives

if $f_{xy}(x,y)$ and $f_{yx}(x,y)$ is continous at $P(x_0,y_0)$.

then $f_{xy}(a,b) = f_{yx}(a,b)$

let $F(\Delta x,\Delta y) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0) - f(x_0, y_0+\Delta y) + f(x_0,y_0)$

$\varphi(x) = f(x, y_0+\Delta y) - f(x, y_0)$

$F(\Delta x,\Delta y) = \varphi(x_0+\Delta x) - \varphi(x_0)$

$\exists \theta_1,\theta_2 \in (0,1)$ $F(\Delta x,\Delta y) = f_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)\Delta x\Delta y$

$\psi(y) = f(x_0+\Delta x, y) - f(x_0, y)$

$F(\Delta x,\Delta y) = \psi(y_0+\Delta y) - \psi(y_0)$

$\Rightarrow F(\Delta x,\Delta y) = f_{yx}(x_0+\theta_3\Delta x, y_0+\theta_4\Delta y)\Delta x\Delta y$

$\Delta x\to 0, \Delta y\to 0 \Rightarrow f_{xy}(x_0,y_0) = f_{yx}(y_0,x_0)$

·复合&隐函数求导 $F(x,y) \Rightarrow \frac{\partial x}{\partial y} = -\frac{F'_y}{F'_x}$

[e.g.1] $\phi(cx-az, cy-bz)=0$ $z=f(x,y)$ 证满足 $a\frac{\partial z}{\partial x} - b\frac{\partial z}{\partial y} = c$

$F(x,y,z) = \phi(\underbrace{cx-az}_{1}, \underbrace{cy-bz}_{2})$ $\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} = -\frac{\phi_1' \cdot c}{\phi_1'\cdot(-a)+\phi_2'(-b)}$ $\frac{\partial z}{\partial y} = -\frac{\phi_2' c}{\phi_1'(-a)+\phi_2'(-b)}$

$\Rightarrow a\frac{\partial z}{\partial x} - b\frac{\partial z}{\partial y} = c$

[e.g.2] $z=z(x,y)$ 由 $xe^x - ye^y = ze^z$ 确定，求dz.

① $F(x,y,z) = xe^x - ye^y - ze^z$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{(x+1)e^x}{(z+1)e^z}$ $\frac{\partial z}{\partial y} = -\frac{(y+1)e^y}{(z+1)e^z}$

$\Rightarrow dz = \frac{(x+1)e^x}{(z+1)e^z}dx - \frac{(y+1)e^y}{(z+1)e^z}dy$

② $d(xe^x - ye^y) = d(ze^z)$

$(e^x+xe^x)dx - (ye^y+e^y)dy = (ze^z+e^z)dz$

$dz = \frac{(x+1)e^x dx - (y+1)e^y dy}{(z+1)e^z}$

☆ 隐函数存在唯一性定理

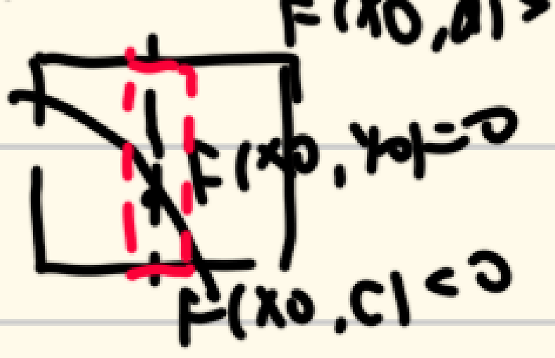
$F(x,y)$ 满足: ① F 以 $P_0(x_0,y_0)$ 为内点的某一区域 $D\subset R^2$ 连续

② $F(x_0,y_0)=0$ ③ 有连续 $F_y(x,y)$ ④ $F_y(x_0,y_0)\neq 0$

则存在一个包含(x_0,y_0)的开矩形 $I\times J\subset D$ 使得:

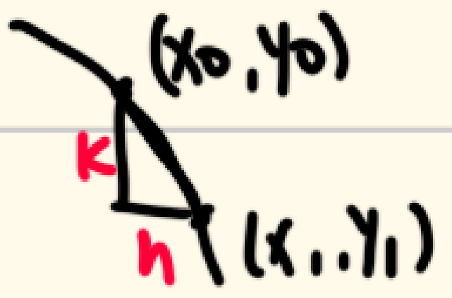
① $x\in I$ $F(x,y)=0$ 确定唯一 $f(x)\in J$ ② $y_0=f(x_0)$ ③ $f\in C^1(I)$ $f'(x) = -\frac{F'_x}{F'_y}$

Proof: let $F_y'(x_0,y_0)>0 \Rightarrow$ exist a small interval $F_y'(x,y)>0$



⇐ a for continuous function, exist $y_0=f(x_0)$ 局部存在

③连续 1° at x_0 $|x-x_0|<\delta \Rightarrow |f(x)-y_0|<\varepsilon$ [收缩 x_0]

2° 

$y=g(x)$ $\begin{cases} F(x_1,y_1)=0 \\ F_y'(x_1,y_1)\neq 0 \end{cases}$ g is continuous at x_1

the function is unique $\Rightarrow f=g$ $\Rightarrow f$ is continuous at $x_1 \in I$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{k}{h}$$ F is differitiable

$\Rightarrow F(x+h,y+k)-F(x,y)=F_x'h+F_y'k+\alpha h+\beta k$ ($\alpha\to 0,\beta\to 0$ when $h\to 0, k\to 0$)

$0-0=F_x'+F_y'\frac{k}{h}+\alpha+\beta\frac{k}{h}$ (when $h\to 0, k\to 0, \alpha\to 0, \beta\to 0$)

$f'(x)=\lim_{h\to 0}\left(-\frac{F_x'+\alpha}{F_y'+\beta}\right)=-\frac{F_x'}{F_y'}$

- The derivative of f at $P_0(x_0,y_0)$ in the direction of the unit vector $u=u_1\vec{i}+u_2\vec{j}$ is

$$\left(\frac{df}{ds}\right)_{u,P_0}=\lim_{s\to 0}\frac{f(x_0+su_1,y_0+su_2)-f(x_0,y_0)}{s}$$ $(D_u f)_{P_0}$

$f(x,y)$ is differentiable at $P(x_0,y_0)\Rightarrow \frac{\partial f}{\partial l}\Big|_{(x_0,y_0)}=f_x\cos\alpha+f_y\sin\alpha$

Proof:

$\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y+o(\sqrt{\Delta x^2+\Delta y^2})$

let $\Delta x=t\cos\alpha$, $\Delta y=t\sin\alpha$ $t=\sqrt{\Delta x^2+\Delta y^2}$

$$\frac{\partial f}{\partial l}\Big|_{(x_0,y_0)}=\lim_{t\to 0^+}\frac{f(x_0+t\cos\alpha,y_0+t\cos\beta)-f(x_0,y_0)}{t}=\lim_{t\to 0^+}\frac{f_x(x_0,y_0)t\cos\alpha+f_y(x_0,y_0)t\sin\alpha+o(t)}{t}$$

$$=f_x(x_0,y_0)\cos\alpha+f_y(x_0,y_0)\sin\alpha$$

- gradient

$$\nabla f(x_0,y_0)=(f_{x_0},f_{y_0})$$

$|e_l|=1$

$$\frac{\partial f}{\partial l}\Big|_{(x_0,y_0)}=\nabla f(x_0,y_0)\cdot e_l=|\nabla f(x_0,y_0)|\cos\langle\nabla f(x_0,y_0),e_l\rangle$$

- Critical point $P(x_0,y_0)$

$f_{xx}(x_0,y_0)=A$ $f_{xy}(x_0,y_0)=B$ $f_{yy}(x_0,y_0)=C$

① $AC-B^2>0$ $\begin{cases} A>0 & \text{extreme minimum point} \\ A<0 & \text{extreme maximum point} \end{cases}$

② $AC-B^2<0$ saddle point

③ $AC-B^2=0$ fail to test

Proof:

consider $P_0(a,b)$ satisfies $f_x = f_y = 0$

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + R_1(x,y)$$
$$= f(a,b) + R_1(x,y)$$

$$f(x,y) - f(a,b) = \frac{1}{2}\left[f_{xx}(x-a)^2 + 2f_{xy}(x-a)(y-b) + f_{yy}(y-b)^2\right]$$

let $h = x-a$, $k = y-b$

$$2[f(x,y) - f(a,b)] = f_{xx}h^2 + 2f_{xy}kh + f_{yy}k^2$$
$$= f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}}\right)k^2$$

$f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{H}{f_{xx}}k^2$

$[f_{xx}f_{yy} - f_{xy}^2 = H]$

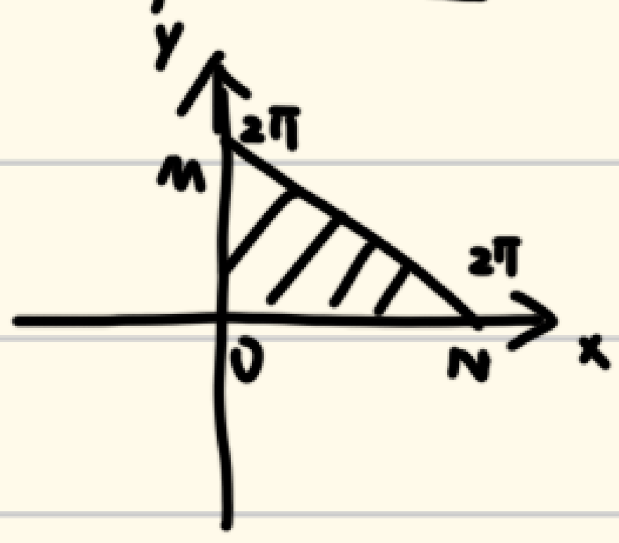
① $f_{xx} > 0$, $H > 0$ $f(x,y) \geq f(a,b)$ ⇒ (a,b) local minimum

② $f_{xx} < 0$, $H > 0$ $f(x,y) \leq f(a,b)$ ⇒ (a,b) local maximum

③ $H < 0$ saddle point

④ $H = 0$ fail to test $[z = x^2+y^2 \,/\, z = x^3+y^3 \,/\, z = -(x^2+y^2)]$

[e.g.1] $f(x,y) = \sin x + \sin y - \sin(x+y)$ 在 x, y轴, $x+y=2\pi$ 围成的闭区域D上的最大/小值



$$\begin{cases} f_x(x,y) = \cos x - \cos(x+y) = 0 \\ f_y(x,y) = \cos y - \cos(x+y) = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{2}{3}\pi \\ y = \frac{2}{3}\pi \end{cases}$$

$f_{xx} = -\sqrt{3}$, $f_{xy} = -\frac{\sqrt{3}}{2}$, $f_{yy} = -\sqrt{3}$ $AC^2 - B > 0$, $A < 0$

$f(\frac{2}{3}\pi, \frac{2}{3}\pi) = \frac{3}{2}\sqrt{2}$ local maximum

边界
$$\begin{cases} OM & x=0 & f(0,y)=0 \\ ON & y=0 & f(x,0)=0 \\ MN & y=2\pi-x & f(x,2\pi-x)=0 \end{cases} \Rightarrow \max = \frac{3}{2}\sqrt{2}, \quad \min = 0$$

- Lagrange Multiplier Method [约束条件为等式]

[用等高线思考. 相切时最优]

$$\begin{cases} \nabla f(\vec{x}) = \lambda \nabla c(\vec{x}) \quad \neq 0 \\ c(\vec{x}) = 0 \end{cases}$$

$\nabla f \parallel \nabla c$ 可行方向 $\perp \nabla c$

$f(\vec{x})$ 沿可行方向的导数为0

$\nabla c_1 \cdots\cdots \nabla c_n$ 与 ∇f 垂直 → $\nabla f \in \mathrm{span}\{\nabla c_1, \cdots \nabla c_n\}$

⇒ $f, c \in C'$. $f(\vec{x})$ s.t. $c(\vec{x}) = 0$ 在 $\vec{x}^*$ 处取到极值 且 $\nabla c(\vec{x}^*) \neq 0$ $\nabla c_1(\vec{x}^*) \cdots \nabla c_n(\vec{x}^*)$ 线性无关, 则 $\exists \lambda \in \mathbb{R}$, s.t. $\begin{cases} \nabla f(\vec{x}^*) = \lambda \nabla c(\vec{x}^*) \\ c(\vec{x}^*) = 0 \end{cases}$ $\nabla f(\vec{x}^*) = \sum_{i=1}^{n} \lambda_i \nabla c_i(\vec{x}^*)$

[e.g.] 求圆内接长方形的最大面积

$S_{max} = 4xy \quad . \quad x^2+y^2=R^2$

$\nabla f = (4y, 4x) = \lambda(2x, 2y)$

$\dfrac{4y}{2x} = \dfrac{4x}{2y} \Rightarrow \dfrac{x}{y} = 1$

$\Rightarrow S_{max} = 4 \cdot \frac{1}{2}R^2 = 2R^2$

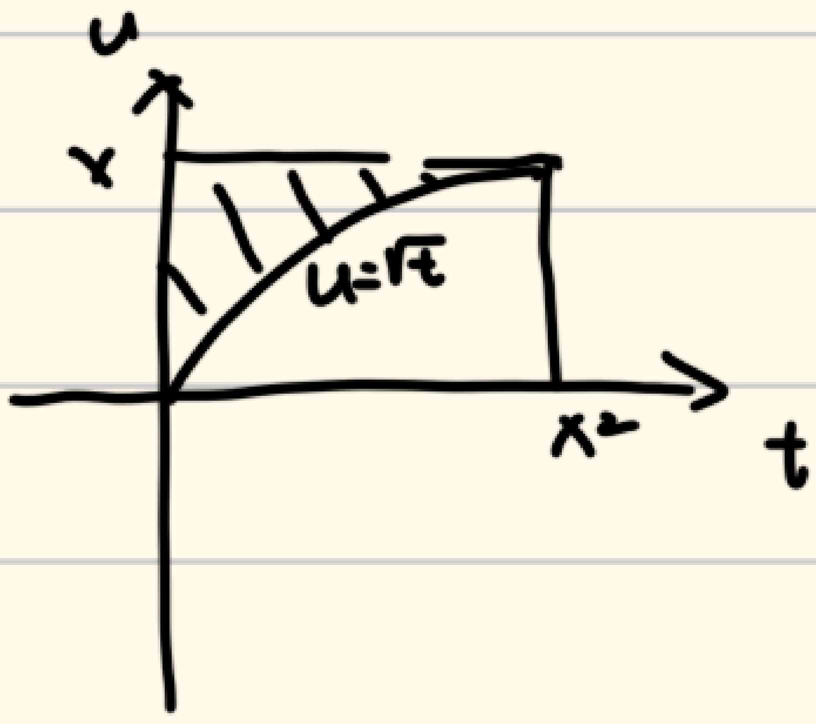
· Talor

$$f(a+h, b+k) = f(a,b) \cdot \sum_{k=1}^{n} \frac{1}{k!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^k + R_n$$

$$R_n(h,k) = \frac{1}{(n+1)!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^{n+1} f\left(a+c(x-a), b+c(y-b)\right) \quad c \in (0,1)$$

· Multiple integrals

[e.g.1] $f(0,0) = -1 \quad . \quad \lim\limits_{x\to 0^+} \dfrac{\int_0^{x^2} dt \int_x^{\sqrt{t}} f(t,u)\,du}{1-e^{-x^3}}$



$$= \lim_{x\to 0^+} \frac{-\int_0^x du \int_0^{u^2} f(t,u)\,dt}{x^3} \qquad (1-e^{-x^3} \sim x^3)$$

$$\overset{L.H.}{=} \lim_{x\to 0^+} \frac{-\int_0^{x^2} f(t,x)\,dt}{3x^2} = -\lim_{x\to 0^+} \frac{x^2 f(\xi, x)}{3x^2} \quad (0<\xi<x^2)$$

$$= -\frac{1}{3}f(0,0) = \frac{1}{3}$$

[e.g.2]

$D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ 具有二阶连续偏导数，在D边界上取零值，且在D上有 $\left|\dfrac{\partial^2 f}{\partial x \partial y}\right| \le M$. Prove: $\left|\iint\limits_D f(x,y)\,dxdy\right| \le \dfrac{M}{4}$

Sol 1 $\quad f(x,0) = f(x,1) = f(0,y) = f(1,y) = 0 \qquad f'_y(0,y) = f'_y(1,y) = 0$

$$\iint\limits_D f(x,y)\,dxdy = \int_0^1 \left[y f(x,y)\Big|_0^1 - \int_0^1 y \frac{\partial f}{\partial y} dy \right] dx$$

$$= \int_0^1 y\,dy \int_0^1 \frac{\partial f}{\partial y}\, d(1-x) \quad \text{简便计算}$$

$$= \int_0^1 y \left[(1-x)\frac{\partial f}{\partial y}\Big|_0^1 - \int_0^1 (1-x)\frac{\partial^2 f}{\partial x \partial y} dx \right] dy$$

$$= -\iint\limits_D y(1-x)\frac{\partial^2 f}{\partial x \partial y}\,dxdy$$

$$\left|\iint\limits_D f(x,y)\,dxdy\right| \le \iint\limits_D \left| y(1-x)\frac{\partial^2 f}{\partial x \partial y}\right| dxdy \le M \iint\limits_D y(1-x)\,dxdy = \frac{M}{4}$$

Sol 2 $\quad f(x,y) = f(x,y) - f(0,y)$

$$= x f'_x(\xi, y) = x\left[f'_x(\xi,y) - f'_x(\xi,0)\right]$$

$$= xy f''_{xy}(\xi, \eta)$$

$$\left|\iint\limits_D f(x,y)\,dxdy\right| \le \iint\limits_D \left| xy f''_{xy}(\xi,\eta)\right| dxdy \le M \iint\limits_D xy\,dxdy = \frac{M}{4}$$

[eg.3]. 一重积分<->二重积分

<1> $\int_0^{\infty}\frac{e^{-ax}-e^{-bx}}{x}dx$ $\quad\int_a^b e^{-xy}dy=\frac{e^{-xy}}{-x}\Big|_a^b$

$=\int_0^{\infty}\int_a^b e^{-xy}dydx=\int_a^b dy\int_0^{\infty}e^{-xy}dx=\int_a^b\frac{1}{y}dy=\ln\frac{b}{a}$

<2> $\int_0^1\frac{x^b-x^a}{\ln x}dx\ (a,b>0)$ $\quad\int_a^b x^y dy=\frac{x^y}{\ln x}\Big|_a^b$

$=\int_0^1\int_a^b x^y dydx=\int_a^b dy\int_0^1 x^y dx$

$=\int_a^b\frac{1}{y+1}dy=\ln\frac{b+1}{a+1}$

<3> $\int_0^{\infty}e^{-x^2}dx$

$D_1=\{(x,y)\mid x^2+y^2\le R^2,\ x\ge 0,\ y\ge 0\}$ $\quad D_2=\{(x,y)\mid x^2+y^2\le 2R^2,\ x\ge 0,\ y\ge 0\}$

$S=\{(x,y)\mid 0\le x\le R,\ 0\le y\le R\}$

$\iint_{D_1}e^{-x^2-y^2}dxdy<\iint_S e^{-x^2-y^2}dxdy<\iint_{D_2}e^{-x^2-y^2}dxdy$

$\iint_S e^{-x^2-y^2}dxdy=\left(\int_0^R e^{-x^2}dx\right)^2$

$\iint_D e^{-x^2-y^2}dxdy=\int_0^{2\pi}\int_0^{\rho}e^{-\rho^2}\rho\,d\rho\,d\theta=\pi(1-e^{-\rho^2})\Rightarrow\begin{cases}D_1=\frac{\pi}{4}(1-e^{-R^2})\\ D_2=\frac{\pi}{4}(1-e^{-2R^2})\end{cases}$

$\lim_{\rho\to\infty}D_1=\lim_{\rho\to\infty}D_2=\frac{\pi}{4}$

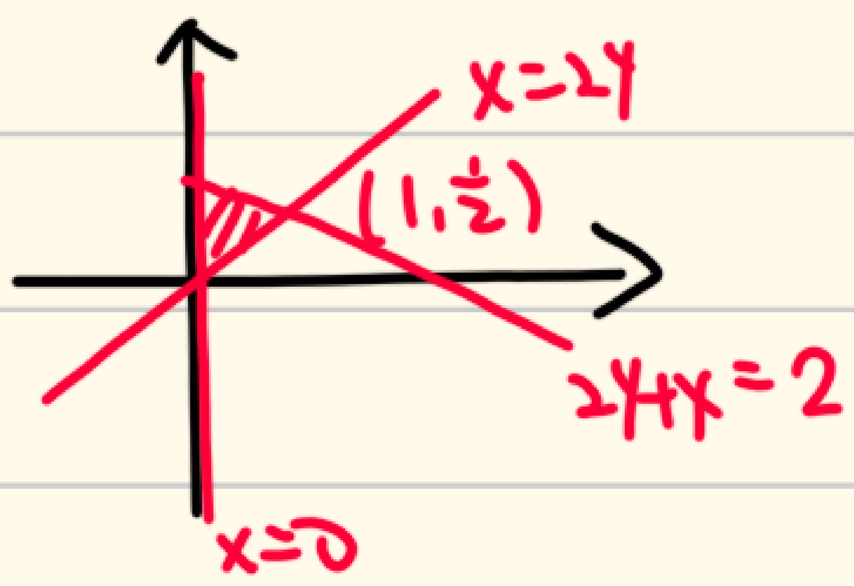
$\Rightarrow\int_0^{\infty}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

3. triple intergal

① 投影法（先一后二）　　② 截面法（先二后一）

[e.g.] find the volume of the solid E bounded by surfaces $x+2y+z=2$, $x=2y$, $x=0$ and $z=0$

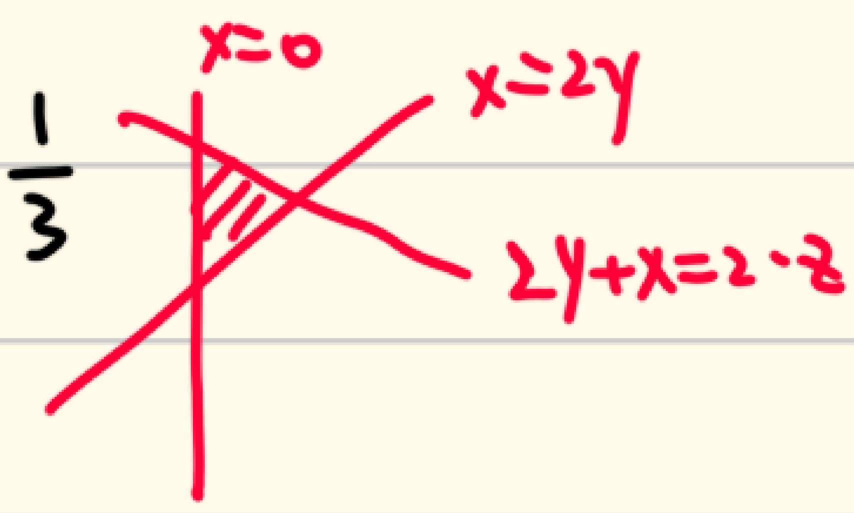
• 投影至 xy-plane



$$V=\iint_D dxdy\int_0^{2-x-2y}dz=\int_0^1\int_{\frac{x}{2}}^{\frac{2-x}{2}}(2-x-2y)\,dydx=\frac{1}{3}$$

• 固定 z

$$V=\int_0^2\iint_{D_z}dxdy\,dz=\int_0^2\int_0^{\frac{2-z}{2}}\int_{\frac{x}{2}}^{\frac{2-x-z}{2}}dydxdz=\frac{1}{3}$$



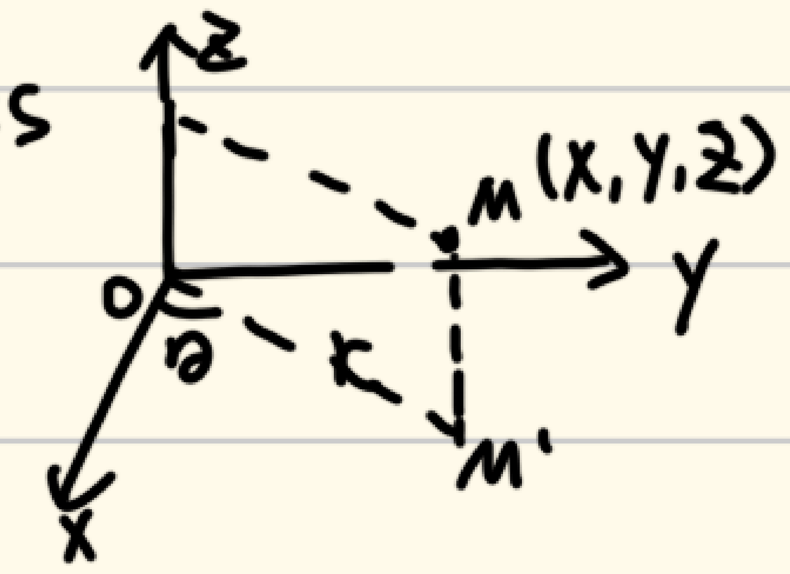
3. polar form

① $D=\{(r,\theta)\mid a\le r\le b,\ \alpha\le\theta\le\beta\}$

$$\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_a^b f(r\cos\theta,r\sin\theta)\,\underline{r}\,drd\theta$$

② cylindrical coordinates

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=z\end{cases}$$



$dv=rdrd\theta dz$

[e.g.] $\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^2(x^2+y^2)\,dzdydx$

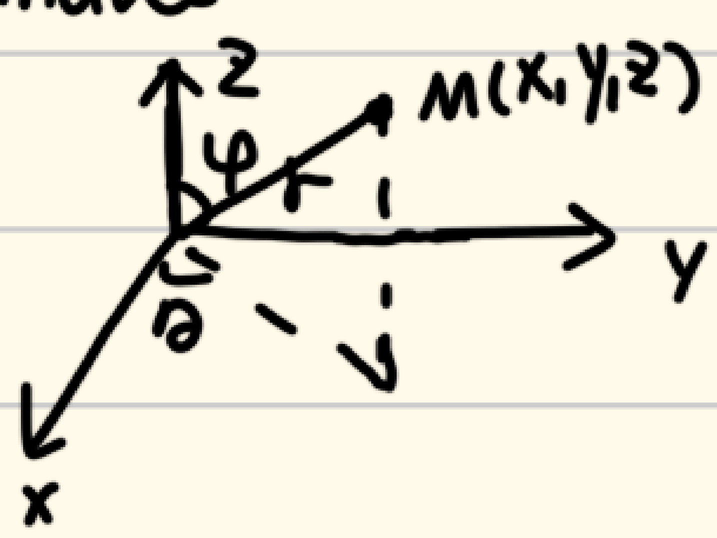
$=\int_0^2\int_0^{2\pi}\int_r^2 r^2\cdot r\,dzd\theta dr$

$=\int_0^2\int_0^{2\pi}(2r^3-r^4)\,d\theta dr$

$=2\pi\left(\frac{1}{2}r^4\Big|_0^2-\frac{1}{5}r^5\Big|_0^2\right)=2\pi\left(8-\frac{32}{5}\right)=\frac{16}{5}\pi$

③ spherical coordinates

$$\begin{cases}x=r\sin\varphi\cos\theta\\ y=r\sin\varphi\sin\theta\\ z=r\cos\varphi\end{cases}$$



$dv=r^2\sin\varphi\,drd\theta d\varphi$

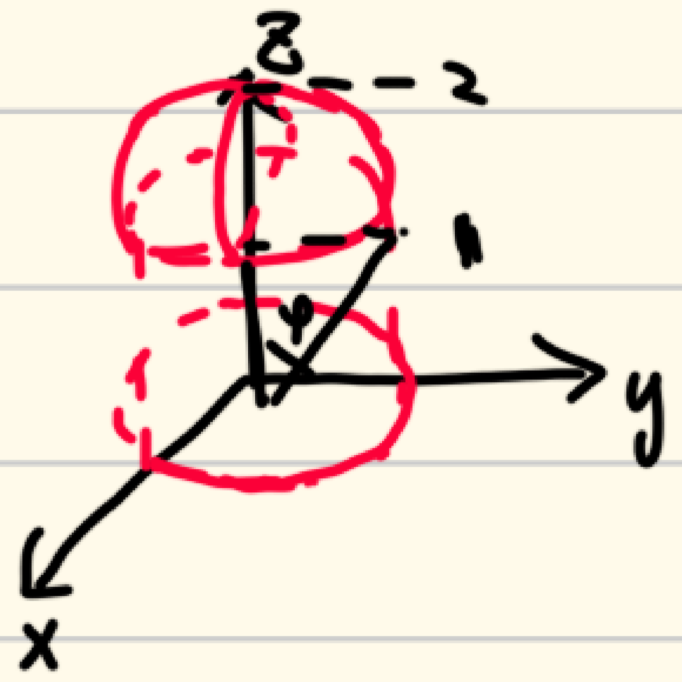
$0\le\theta\le2\pi,\ 0\le\varphi\le\pi$

[e.g.] find the volume of solid E bounded by $x^2+y^2\le 1$ and $x^2+y^2+z^2\le 4$

$$\begin{cases} x^2+y^2\le 1 \Rightarrow r\sin\varphi \le 1 \Rightarrow r\le \cot\varphi \\ x^2+y^2+z^2\le 4 \Rightarrow r\le 2 \end{cases}$$

$$V=\int_0^{2\pi}\left(\int_0^{\frac{\pi}{6}}\int_0^{2}+\int_{\frac{5}{6}\pi}^{\pi}\int_0^{2}+\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi}\int_0^{\cot\varphi}\right)r^2\sin\phi\, dr\,d\phi\,d\theta=\frac{4}{3}\pi(8-3\sqrt{3})$$

[e.g.] $I=\int_{-1}^{1}dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\int_{1}^{1+\sqrt{1-x^2-y^2}}\frac{dz}{\sqrt{x^2+y^2+z^2}}$



$z=1+\sqrt{1-x^2-y^2} \Rightarrow x^2+y^2+(z-1)^2=1$

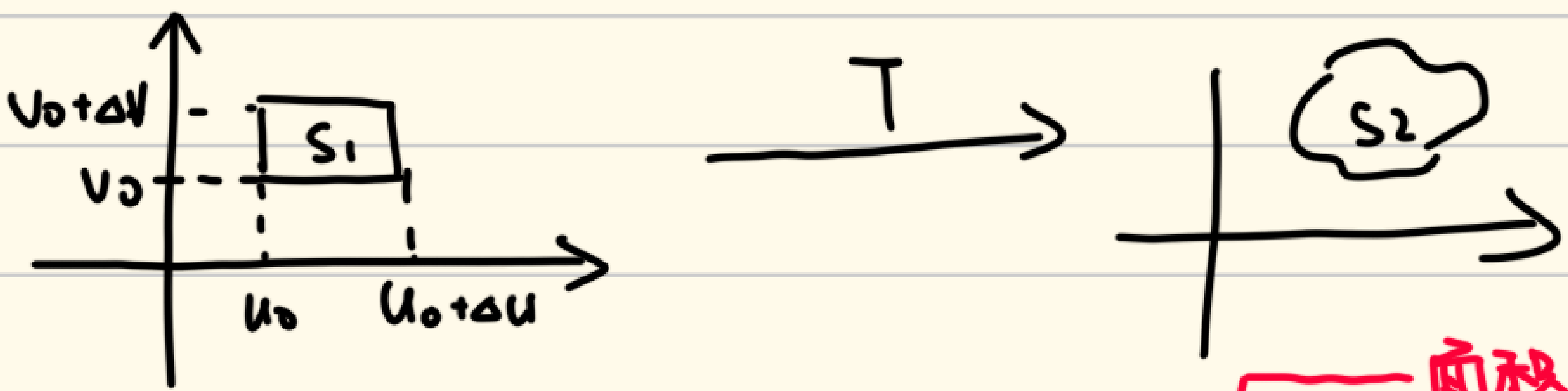
$$\begin{cases} z=1 \Rightarrow r=\frac{1}{\cos\varphi} \\ x^2+y^2+z^2=2z \Rightarrow r=2\cos\varphi \end{cases}$$

$$I=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_{\sec\varphi}^{2\cos\varphi}\frac{1}{r}\cdot r^2\sin\varphi\, dr$$

$$=\frac{\pi}{3}(7-4\sqrt{2})$$

4. substitution [Jacobian 雅可比矩阵]

若$T(u,v)=(x(u,v),y(u,v))$ 且 $x(u,v)$, $y(u,v)$ 有偏导，则 $J_T=\begin{pmatrix} x_u' & x_v' \\ y_u' & y_v' \end{pmatrix}$



$S_2=|J_T|\,S_1$

面积不像一维的情况需要考虑方向性

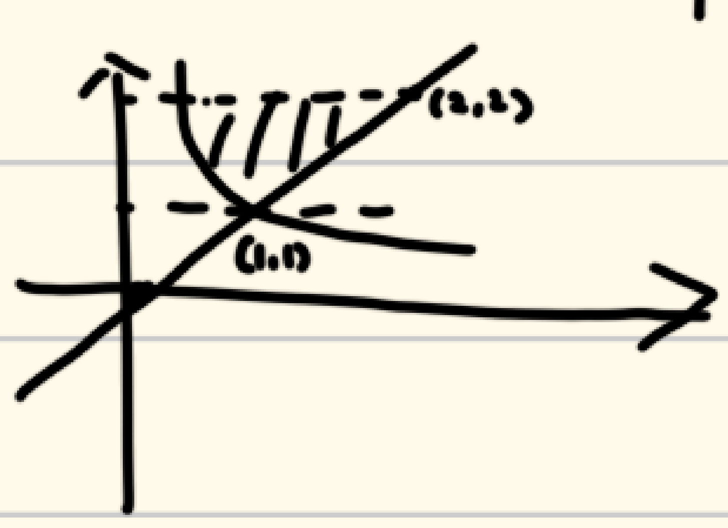
$$\iint_D f(x,y)\,dxdy=\iint f(x(u,v),y(u,v))\,\underbrace{|J_T|\,dudv}_{dxdy}$$

☆局部线性映射 [f在(x_0,y_0)处最佳线性映射所近似的矩阵]

[e.g.] $\int_1^2\int_{\frac{1}{y}}^{y}\sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dxdy$

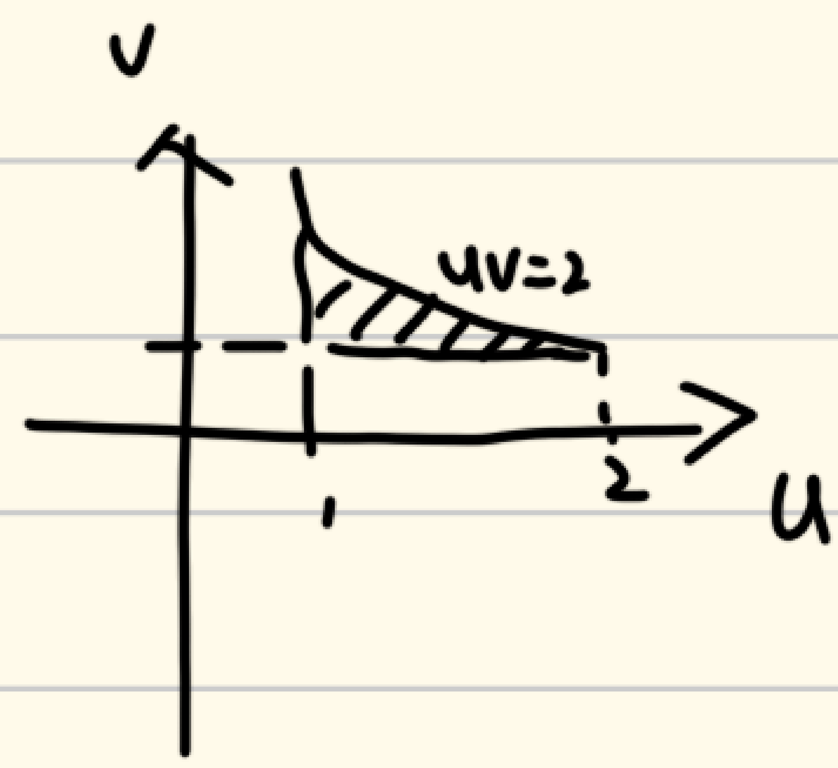
let $u=\sqrt{xy}$, $v=\sqrt{\frac{y}{x}}$ $\quad f(u,v)=ve^u \quad \Rightarrow \begin{cases} x=\frac{u}{v} \\ y=uv \end{cases}$

$$|J|=\begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}=\frac{u}{v}+\frac{u}{v}=\frac{2u}{v}$$



$y=2 \Rightarrow uv=2$
$y=x \Rightarrow v=1$
$xy=1 \Rightarrow u=1$

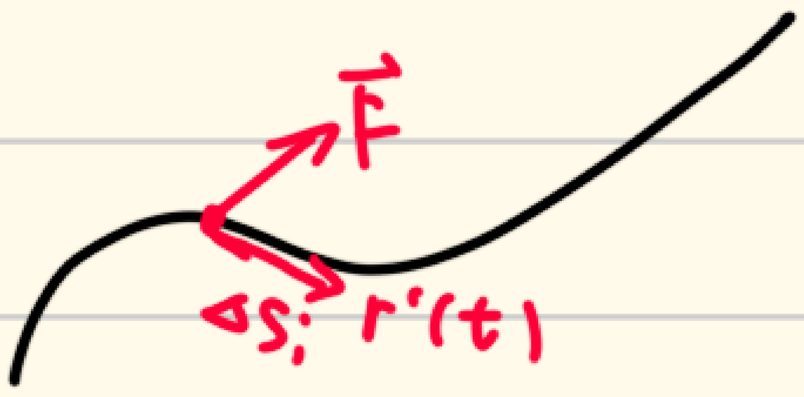
$\Rightarrow$



$$V=\int_1^2\int_1^{\frac{2}{u}}|J|\cdot ve^u\,dvdu=\int_1^2\int_1^{\frac{2}{u}}2ue^u\,dvdu=\int_1^2 4e^u-2ue^u\,du=2e^2-4e$$

· vector fields and line integrals 线面积分

1. work (W)



$$\Delta W_i = \vec{F} \cdot \frac{r'(t)}{|r'(t)|} \cdot \Delta S_i$$

$$W = \int_C \vec{F} \cdot \frac{r'(t)}{|r'(t)|} ds = \int_C \vec{F} \cdot \vec{T} ds = \int_C \vec{F} \cdot \frac{r'(t)}{|r'(t)|} \cdot |r'(t)| dt$$

$$= \int_a^b \vec{F} \cdot r'(t) dt$$

[e.g.] $\vec{F} = (x, y, z)$, $\vec{r}(t) = (\cos(\pi t), t^2, \sin(\pi t))$ $0 \le t \le 1$

$$W = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F} \cdot r'(t) dt = \int_0^1 (\cos(\pi t), t^2, \sin(\pi t)) (-\pi\sin(\pi t), 2t, \pi\cos(\pi t)) dt$$

$$= \int_0^1 2t^3 = \frac{1}{2}$$

2. Circulation and Flux [通量与环通量]

$\uparrow N$ $\rightarrow M$

C is simple and closed, we have $\oint_C \vec{F} \cdot \vec{T} ds = \int_a^b M dx + N dy$ (counter-clockwise is the positive direction)

Flux: $\int_C \vec{F} \cdot \vec{n} ds$ $\quad \vec{n} ds = (dy, -dx)$

$$\Rightarrow \oint_C \vec{F} \cdot \vec{n} ds = \int_a^b M dy - N dx$$

[e.g.] Find the circulation and flux of the velocity field $\vec{F} = (x-y, x)$ of the unit circle.

$C = (\cos t, \sin t)$ $\quad \vec{T} = (-\sin t, \cos t)$ $\quad \vec{n} = (\cos t, \sin t)$ $\quad \vec{F} = (\cos t - \sin t, \cos t)$

$$\oint_C \vec{F} \cdot \vec{T} ds = \int_0^{2\pi} (1 - \tfrac{1}{2}\sin 2t) dt = 2\pi$$

$$\oint_C \vec{F} \cdot \vec{n} ds = \int_0^{2\pi} \cos^2 t \, dt = \pi$$

3. conservative field 保守场

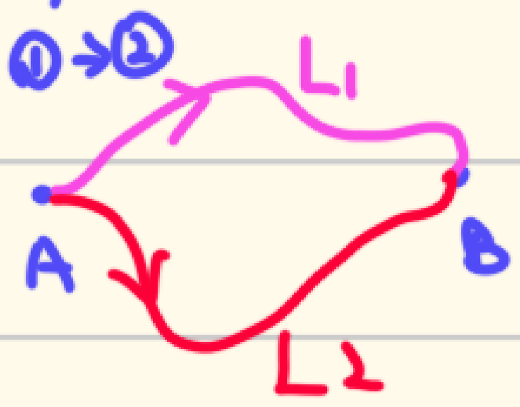
$V \in \mathbb{R}^n$

① $\oint_L \vec{F} dr = 0$ ($\forall$ 闭合光滑曲线 $\in V$ 成立)

② $\forall A, B \in V$ $\int_L \vec{F} d\vec{r}$ 与路径无关 $\quad \bigstar \ \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

proof:

<1> ① → ②



$$\oint_{L_1 \cup (-L_2)} \vec{F} d\vec{r} = 0 = \int_{L_1} \vec{F} d\vec{r} - \int_{L_2} \vec{F} d\vec{r} = 0 \Rightarrow \int_{L_1} \vec{F} d\vec{r} = \int_{L_2} \vec{F} d\vec{r} = 0$$

<2> ② → ①

$$\int_{L_1} \vec{F} d\vec{r} = \int_{L_2} \vec{F} d\vec{r} \qquad \oint_{F} \vec{F} d\vec{r} = \oint_{(-L_1) \cup L_2} \vec{F} d\vec{r} = -\int_{L_1} \vec{F} d\vec{r} + \int_{L_2} \vec{F} d\vec{r} = 0$$

$du = U_x dx + U_y dy$, $Q_x = P_y \Leftarrow$ $P_y = U_{xy} = Q_x = U_{yx}$

③ $\vec{F}$ is conservative on D iff $\vec{F}$ is a gradient field ($\vec{F} = \nabla u$ for some differentiable function u.)

Proof

<1> $\vec{F}$ is a gradient field

$$\oint_{\vec{r}} \vec{F}\, d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\, \vec{r}'(t)\, dt = \int_a^b \left(\nabla \varphi(\vec{r}(t))\right) \vec{r}'(t)\, dt = \int_a^b [\varphi(\vec{r}(t))]'\, dt$$

$$= \varphi[\vec{r}(t)]_a^b = \varphi(\vec{r}(b)) - \varphi(\vec{r}(a))$$

<2> $\vec{F}$ is conservative

$F(x,y) = (P, Q)$ $\quad (x_0, y_0) \in V$, $\varphi(x,y) = \int_{(x_0,y_0)}^{(x,y)} \vec{F}\, d\vec{r}$ $\quad (x,y) \in V$

$$\varphi_x' = \lim_{h \to 0} \frac{\varphi(x+h, y) - \varphi(x,y)}{h} = \lim_{h \to 0} \frac{\int_{(x,y)}^{(x+h,y)} \vec{F}\, d\vec{r}}{h}$$

let $\vec{r}_2(t) = (t, y)$ $\quad t \in [x, x+h]$ $\quad \vec{r}_2{}'(t) = (1, 0)$

y: constant

$$\Rightarrow \varphi_x' = \lim_{h \to 0} \frac{\int_{(x,y)}^{(x+h,y)} \vec{F}\, d\vec{r}}{h} = \frac{\int_x^{x+h} \vec{F}(\vec{r}_2(t))\, \vec{r}_2{}'(t)\, dt}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} (P(t,y), Q(t,y)) \cdot (1,0)\, dt}{h}$$

$$= \lim_{h \to 0} \frac{\int_x^{x+h} P(t,y)\, dt}{h} \overset{L.H}{=} \lim_{h \to 0} \frac{P(x+h,y) \cdot (x+h)' - P(x,y) \cdot x'}{h'} = \lim_{h \to 0} P(x+h, y) = P(x,y)$$

Similarly, $\varphi_y' = Q(x,y)$

$$\Rightarrow \vec{F} = \nabla \varphi = (\varphi_x, \varphi_y)$$

[e.g.] Show that the vector field defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ is not conservative:

$$\vec{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

Choose loop C: $x^2 + y^2 = 1$ $\quad \vec{F} = (-\sin t, \cos t)$ $\quad d\vec{r} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) dt = (-\sin t, \cos t)\, dt$

$$\oint_C \vec{F}\, d\vec{r} = \int_0^{2\pi} (-\sin t, \cos t)^2\, dt = 2\pi \neq 0$$ $\Rightarrow$ not conservative

4. surface and area

- Smooth surface : if $\vec{r}$ is differentiable with continuous partial derivatives $\vec{r}_u$ and $\vec{r}_v$, and the cross product $\vec{r}_u \times \vec{r}_v$ is never 0 vector.

- ① $A(S) = \iint_S d\sigma = \iint_R |\vec{r}_u \times \vec{r}_v| \, du dv$

②* $\vec{r}(x,y) = (x, y, f(x,y))$ $\Rightarrow$ $\boxed{A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1} \, dxdy}$ $\quad z = \sqrt{x^2+y^2} \Rightarrow A(S) = \sqrt{2} \iint_D dxdy$

$\vec{r}_x \times \vec{r}_y = (-f_x, -f_y, 1)$

③ $f(x,y,z) = c$ $\quad u = x, v = y, z = h(u,v)$

$\Rightarrow r(u,v) = u\vec{i} + v\vec{j} + h(u,v)\vec{k}$

$r_u = \vec{i} + \frac{\partial h}{\partial u}\vec{k} = \vec{i} - \frac{f_x}{f_z}\vec{k} \qquad r_v = \vec{j} + \frac{\partial h}{\partial v}\vec{k} = \vec{j} - \frac{f_y}{f_z}\vec{k}$

$|r_u \times r_v| = \left|\frac{1}{f_z}(f_x\vec{i} + f_y\vec{j} + f_z\vec{k})\right| = \left|\frac{\nabla F}{f_z}\right| = \frac{|\nabla F|}{|\nabla F \cdot p|} \qquad (p = \vec{k})$

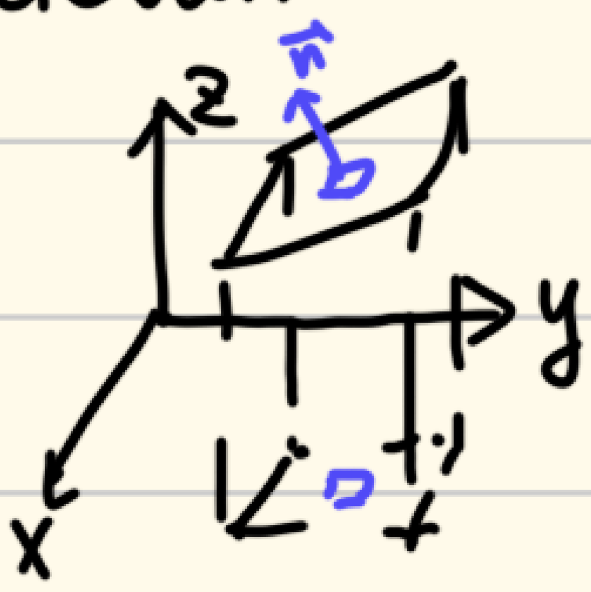
$A(S) = \iint_D \frac{|\nabla F|}{|\nabla F \cdot p|} \, dxdy \qquad$ (p is normal to D and $\nabla F \cdot p \neq 0$)

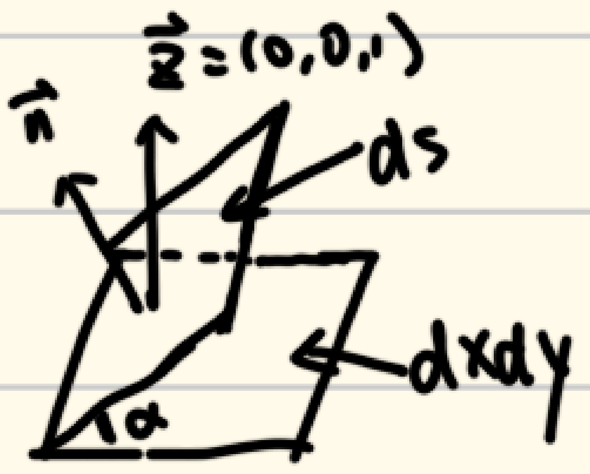
- flux $\iint_S \vec{F} \cdot \vec{n} \, d\sigma$

$\vec{r}(u,v)$: $\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \cdot |\vec{r}_u \times \vec{r}_v| \, dudv$
$= \iint_D \vec{F} (\vec{r}_u \times \vec{r}_v) \, dudv$

$z = f(x,y)$: $\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \iint_D \vec{F}(-f_x, -f_y, 1) \, dxdy$

*detail:





$\cos\alpha = \frac{dxdy}{ds} = |\cos\langle \vec{n}, \vec{z} \rangle|$ $\qquad \vec{F} = (P, Q, R)$

取向上为正

$\vec{n} = (-z_x, -z_y, 1)$

$\Rightarrow \cos\alpha = \frac{|\vec{n} \cdot \vec{z}|}{|\vec{n}| \cdot |\vec{z}|} = \frac{1}{\sqrt{z_x^2 + z_y^2 + 1}} \Rightarrow ds = \sqrt{z_x^2 + z_y^2 + 1} \, dxdy \Rightarrow S = \iint \sqrt{z_x^2 + z_y^2 + 1} \, dxdy$

$\vec{n}_0 \cdot ds = \frac{(-z_x, -z_y, 1)}{\sqrt{z_x^2 + z_y^2 + 1}} \cdot ds = (-z_x, -z_y, 1) dxdy \qquad \iint_\Sigma \vec{F} \cdot \vec{n}_0 \, ds = \iint_D (-P z_x - Q z_y + R) \, dxdy$

[e.g.] $z = \sqrt{x^2+y^2}$ 被 $z^2 = 2y$ 所截部分的面积、

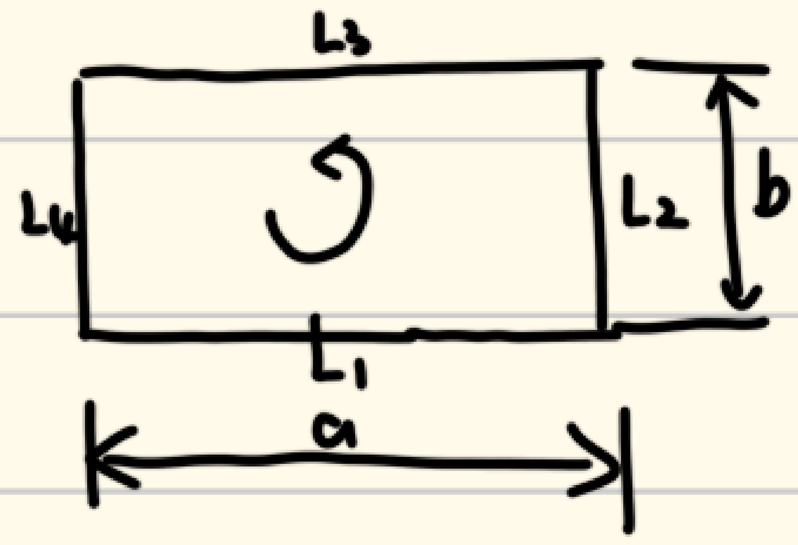
$\begin{cases} z = \sqrt{x^2+y^2} \\ z^2 = 2y \end{cases} \Rightarrow D: x^2 + y^2 = 2y, \text{ i.e. } x^2 + (y-1)^2 = 1$

$z_x^2 + z_y^2 + 1 = \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1 = \sqrt{2}$ $\qquad \Rightarrow S = \iint_D \sqrt{2} \, dxdy = \sqrt{2}\pi$

5. Green's Theorem [逆时针为正向]

$$\oint_C P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

$$\oint_C \vec F\cdot d\vec r = \iint_D |\nabla\times\vec F|\,d\sigma$$



L_1: $\int_{L_1} P\,dx+Q\,dy = \int_{L_1} P(x,0)\,dx$

L_2: $\int_{L_2} Q(a,y)\,dy$ L_3: $\int_{L_3} P(x,b)\,dy$ L_4: $\int_{L_4} Q(0,y)\,dy$

$L_1\cup L_3$: $\int_0^a P(x,0)\,dx+\int_a^0 P(x,b)\,dx=\int_0^a[P(x,0)-P(x,b)]\,dx$

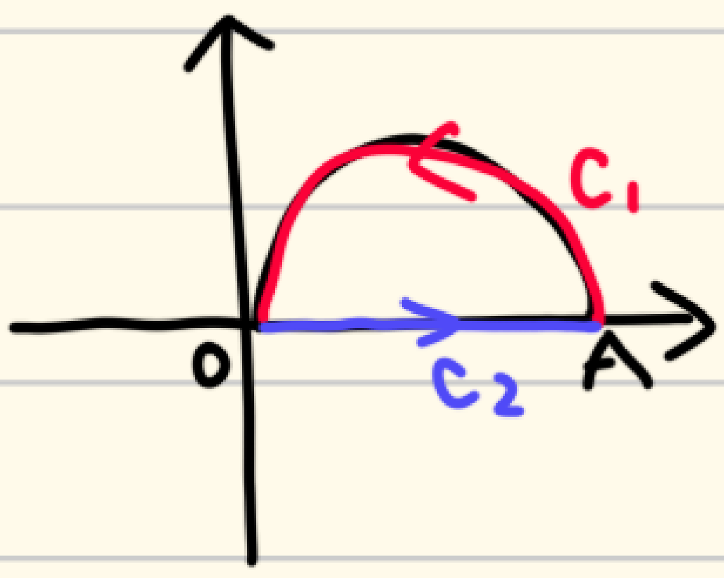
$=\int_0^a P(x,y)\Big|_{y=b}^{y=0}dx = \int_0^a dx\int_b^0 \frac{\partial P}{\partial y}dy = -\int_0^a\int_0^b \frac{\partial P}{\partial y}\,dy\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$ $D=[0,a]\times[0,b]$

$L_2\cup L_4$: $\iint_D \frac{\partial Q}{\partial x}\,dA$ $\vec F=(M,N)$

$\oint_C \vec F\cdot\vec T\,ds = \oint_C M\,dx+N\,dy = \iint_R (\underline{N_x-M_y})\,dA$ circulation density

$\oint_C \vec F\cdot\vec n\,ds = \oint_C -N\,dx+M\,dy = \iint_R (\underline{M_x+N_y})\,dA$ flux density

[e.g.1] $I=\int_{AO}(e^x\sin y-4y+x)\,dx+(e^x\cos y-3)\,dy$ AO 为 $x^2+y^2=ax\ (y\ge 0, a>0)$ $A(a,0)\to O(0,0)$ 部分



$(P,Q)=(e^x\sin y-4y+x,\ e^x\cos y-3)$

$Q_x-P_y=4$

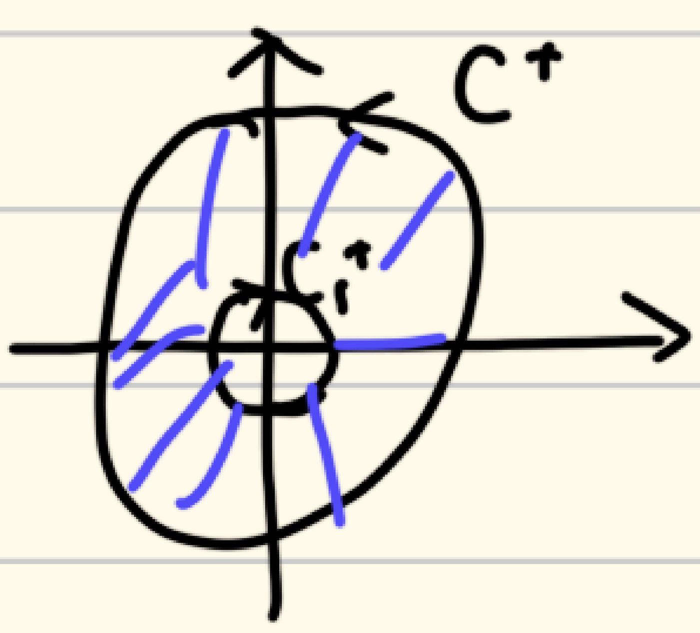
$\int_{AO}=\int_{C_1^+}+\int_{C_2^+}-\int_{C_2^+}=\iint_D 4\,d\sigma-\int_{C_2^+}=\frac12\pi a^2-\int_0^a t\,dt=\frac{\pi-1}{2}a^2$

[e.g.2] $I=\oint_{C^+}\frac{x\,dy-y\,dx}{x^2+y^2}$ C^+是包围原点的任一正向闭合曲线

$\boxed{Q_x-P_y=0}$

☆ 原点无定义

let $\varepsilon\to 0^+$. $C_1^+: x^2+y^2=\varepsilon^2$



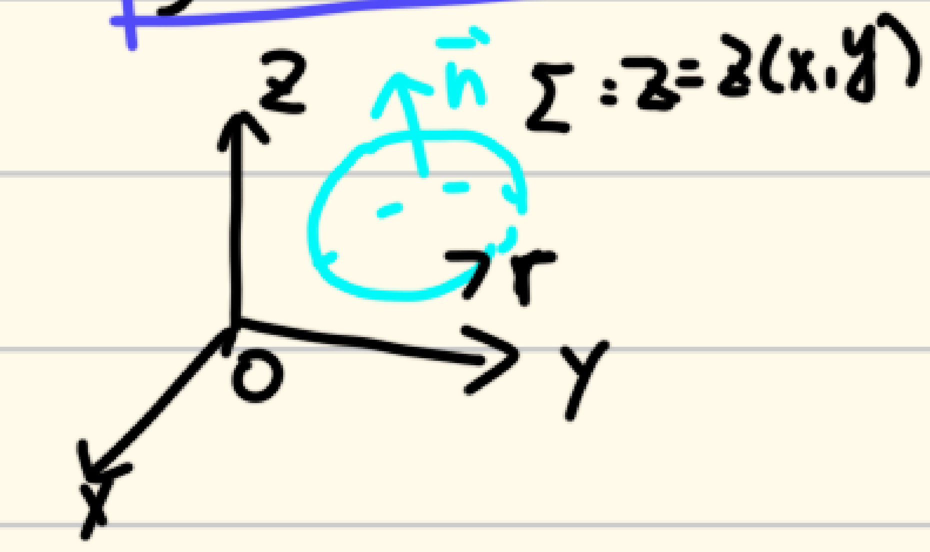
$\int_{C^+}=\int_{C^+}+\int_{C_1^+}-\int_{C_1^+}$ $C_1: \begin{cases}x=\varepsilon\cos\theta\\ y=\varepsilon\sin\theta\end{cases}$

$=0-\int_{C_1^+}$

$=\int_0^{2\pi}\frac{\varepsilon^2(\cos^2\theta+\sin^2\theta)}{\varepsilon^2}\,d\theta=2\pi$

6. Stoke's Theorem　曲线积分 $\Longleftrightarrow$ 曲面积分

$$\boxed{\oint_C F\cdot dr = \iint_S \nabla\times F\cdot n\, dr}$$

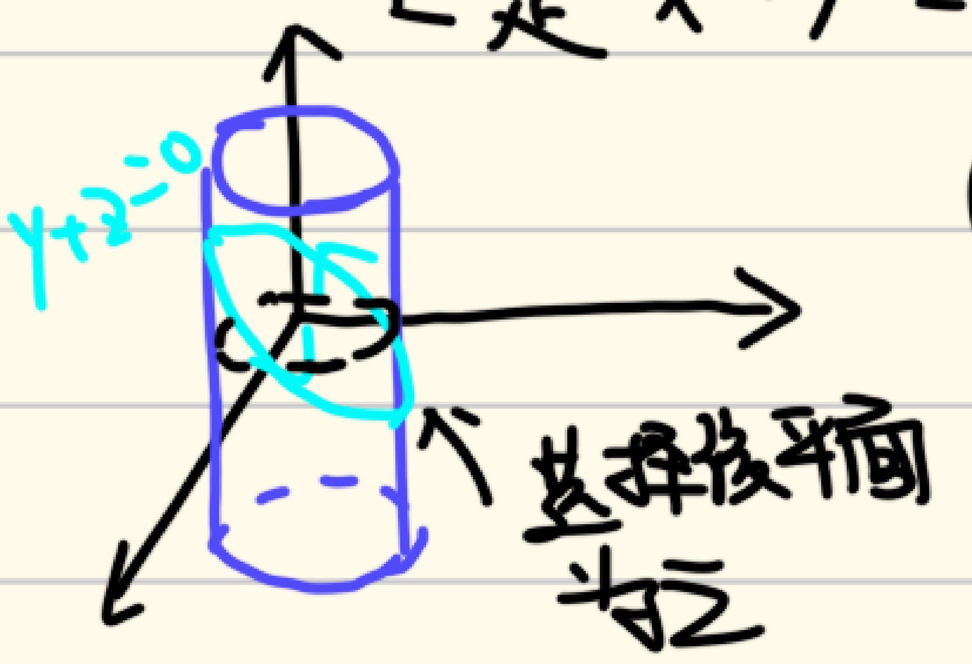


$$\uparrow \quad \iint_\Sigma \begin{vmatrix} dydz & dxdz & dxdy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

$$\boxed{\oint_\Gamma Pdx+Qdy+Rdz = \iint_\Sigma \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz - \left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)dxdz + \left(\frac{\partial Q}{x}-\frac{\partial P}{\partial y}\right)dxdy}$$

$$=\iint_\Sigma \underbrace{\nabla\times F}_{\text{curl } F}\cdot n\, dr$$

[e.g.] L是 $x^2+y^2=1$ 与 $y+z=0$ 交线. 从z轴正向往负向看为逆时针. 求 $\oint_C z\,dx+y\,dz$



$(P,Q,R)=(z,0,y)$

$$I=\iint_\Sigma \overset{P}{1}dydz + \overset{Q}{1}dxdz \qquad R=0$$

$[z=-y]$

$$=\iint_D (-P\underset{0}{z_x} - Q\underset{-1}{z_y} + R)\,dxdy = \pi$$

7. divergence theroem / gauss therom

$$\boxed{\iint_S \vec F\cdot\vec n\, d\sigma = \iiint_D \nabla\cdot\vec F\, dv \qquad \oiint_S Pdydz+Qdxdz+Rdxdy = \iiint_V \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz}$$

flux:  $[x_0,x_0+\Delta x]\times[y_0,y_0+\Delta y]\times[z_0,z_0+\Delta z]$

$$\iint_{Top} \vec F\cdot k\, d\sigma + \iint_{Bottom} \vec F\cdot(-k)\,d\sigma = \int_{x_0}^{x_0+\Delta x}\int_{y_0}^{y_0+\Delta y}\left[P(x,y,z_0+z)-P(x,y,z)\right]dydx$$

$$=\iiint_D P_z\, dV$$

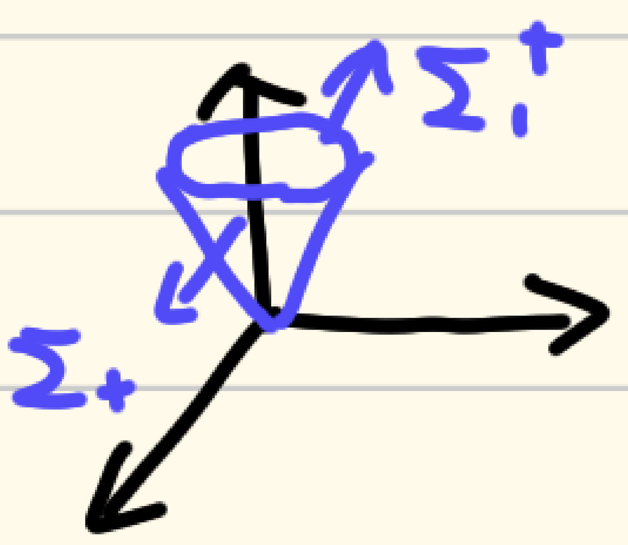
$$\Rightarrow \iiint_D (M_x+N_y+P_z)\,dV = \boxed{\iiint_D \nabla\cdot\vec F\, dv}$$

[e.g.] $I=\iint_\Sigma x\,dydz + 2xy\,dxdz - 2z\,dxdy$　Σ为 $z=\sqrt{x^2+y^2}$ $(0\le z\le h)$ 下侧

$$I=\iint_{\Sigma+} + \iint_{\Sigma_1+} - \iint_{\Sigma_1+} = \iiint_\Omega (1+2x-2)\,dV - \iint_{\Sigma_1+} = \iiint_\Omega (2x-1)\,dv - 2\pi h^3$$

$$=\int_0^{2\pi}\int_0^h\int_0^z (2r\cos\theta-1)\,r\,dr\,dz\,d\theta$$

$$=-\frac{5}{3}\pi h^3$$



$(P,Q,R)=(x,2xy,-2z)$

$\vec n=(0,0,1)$

$$\iint_{\Sigma+} = \iint(\underbrace{-Pz_x-Qz_y+R}_{\vec F\cdot\vec n})\,dxdy = \iint(-2z)\,dxdy = -2\pi h^3$$

取下侧 $\Rightarrow$ 答案为 $\frac{5}{3}\pi h^3$

* $\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]^T = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z$

- 梯度 gradient $\nabla \vec{F} = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]^T$
- 散度 div $\nabla \cdot \vec{f} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]^T \cdot \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$
- 旋度 curl

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \vec{e}_x\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) - \vec{e}_y\left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}\right) + \vec{e}_z\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)$$

对标量场的梯度求 $\begin{cases} \text{散度} & \nabla \cdot \nabla(\varphi) = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \\ \text{旋度} & \nabla \times \nabla(\varphi) = \vec{0} \end{cases}$

对旋度求散度 $\nabla \cdot (\nabla \times \vec{f}) = \vec{0}$