- The definition of the limit

If $f(x)$ is arbitrarily close to L, for all x sufficiently close to c, then

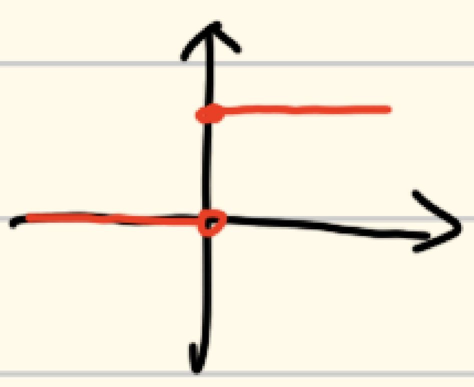
$$\lim_{x\to c} f(x)=L$$

Note that $x\to c$ but $x\neq c \Leftrightarrow L$ has nothing to do with $f(c)$.

$\Leftrightarrow$ if $\forall \varepsilon>0, \exists \delta>0$ such that if $x\in I$, $0<|x-c|<\delta$, then $|f(x)-L|<\varepsilon$

$\hookrightarrow x\neq c$

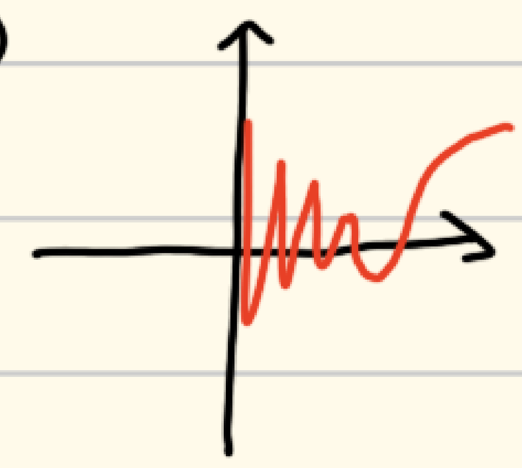
- limit DNE (does not exist)

① 

jump

② 

unbound

③ 

Oscillate

- limit laws

$$\lim_{x\to c}(f(x)\pm g(x))=\lim_{x\to c}f(x)\pm\lim_{x\to c}(g(x))$$

$$\lim_{x\to c}(f(x)\cdot g(x))=\lim_{x\to c}f(x)\cdot\lim_{x\to c}g(x)$$

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)}$$

$$\lim_{x\to c}[f(x)]^n=[\lim_{x\to c}f(x)]^n \quad n\in N$$

$$\lim_{x\to c}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to c}f(x)} \quad n\in N$$

$\longrightarrow$ $P(x)=\sum_{i=0}^{n}a_i x^i$

$$\lim_{x\to c}P(x)=\sum_{i=0}^{n}a_i c^i$$

- The Sandwich Theorem 夹逼定理

Suppose that $g(x)\le f(x)\le h(x)$ $\quad x\in(c-\delta, c+\delta)$, $x\neq c$ and $\delta>0$, $\delta\to 0$

if $\lim_{x\to c}g(x)=\lim_{x\to c}h(x)=L$

then $\lim_{x\to c}f(x)=L$

$\forall\varepsilon>0, \exists\delta_1>0, \delta_2>0$ $\quad |g(x)-A|<\varepsilon$ whenever $0<|x-c|<\delta_1$

$\Leftrightarrow A-\varepsilon<g(x)<A+\varepsilon$ s.t. $A-\varepsilon<h(x)<A+\varepsilon$ $(0<|x-c|<\delta_2)$

let $\delta=\min(\delta_1,\delta_2)$. s.t.

$A-\varepsilon<g(x)\le f(x)\le h(x)<A+\varepsilon$ whenever $0<|x-c|<\delta$

so we can conclude $\lim_{x\to c}f(x)=L$

$\to$ Suppose that $f(x)\le g(x)$ $\quad x\in(c-\delta, c+\delta)$, $x\neq c$ and $\delta>0$, $\delta\to 0$

then $\lim_{x\to c}f(x)\boxed{\le}\lim_{x\to c}g(x)$

- use ε-δ language to prove the limit

① $\lim_{x\to x_0}(f(x)\pm g(x))=\lim_{x\to x_0}f(x)\pm\lim_{x\to x_0}g(x)$

$\forall\varepsilon>0$ $\exists\delta_1,\delta_2>0$, $|f(x)-A|<\frac{\varepsilon}{2}$ whenever $0<|x-x_0|<\delta_1$; $|g(x)-B|<\frac{\varepsilon}{2}$ whenever $0<|x-x_0|<\delta_2$

let $\varepsilon=\min\{\delta_1,\delta_2\}$: we have $|f(x)\pm g(x)-(A\pm B)|=|(f(x)-A)\pm(g(x)-B)|\le|f(x)-A|+|g(x)-B|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

② $\lim_{x\to x_0}[f(x)\cdot g(x)]=\lim_{x\to x_0}f(x)\cdot\lim_{x\to x_0}g(x)$ Given that $|f(x)\cdot g(x)-AB|=|f(x)g(x)-g(x)A+g(x)A-AB|\le|g(x)||f(x)-A|+|A||g(x)-B|$

$\forall\varepsilon>0, \exists\delta_1,\delta_2>0$. $|g(x)|\le M$ whenever $0<|x-x_0|<\delta_1$, $|f(x)-A|<\frac{\varepsilon}{2M}$, $|g(x)-B|<\frac{\varepsilon}{2|A|}$ whenever $0<|x-x_0|<\delta_2$

let $\delta=\min\{\delta_1,\delta_2\}$, when $0<|x-x_0|<\delta$ s.t. $|f(x)g(x)-AB|<M\cdot\frac{\varepsilon}{2M}+|A|\cdot\frac{\varepsilon}{2|A|}=\varepsilon$

③ $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} \iff \lim_{x\to x_0} f(x)\cdot\frac{1}{g(x)} = \lim_{x\to x_0} f(x)\cdot\frac{1}{\lim_{x\to x_0} g(x)}$ [②]

$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \frac{|g(x)-B|}{|g(x)||B|}$, let $0<|g(x)-|B||<\frac{|B|}{2}$ Given that $|g(x)|=|g(x)-B-(-B)| \ge ||g(x)-B|-|B|| \ge \frac{|B|}{2}$

$\left|\frac{1}{g(x)} - \frac{1}{B}\right| < \frac{2}{B^2}|g(x)-B|$ let $\varepsilon = \mathrm{Min}\{\frac{|B|}{2}, \frac{|B|^2}{2}\varepsilon\}$, then $\left|\frac{1}{g(x)} - \frac{1}{B}\right| < \frac{2}{|B|^2}\delta = \varepsilon$

☆ Some formulas

① $|x+y| \le |x| + |y|$

② $||x|-|y|| \le |x-y|$

$\begin{cases} |x| = |(x-y)+y| \le |x-y| + |y| \\ |y| = |(y-x)+x| \le |x-y| + |x| \end{cases}$

③ $a^n - b^n = (a-b)\sum_{i=0}^{n-1} a^i \cdot b^{n-i-1}$

• Continuity at $x=c$ $\begin{cases} x=c \text{ is defined} \\ \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c) \end{cases}$

$\forall \varepsilon > 0.\ \exists \delta > 0$ s.t. $|f(x)-f(c)| < \varepsilon$ whenever $c-\delta < x < c+\delta$

☆ $\lim_{x\to c} f(x) = b$ and g is continuous at the point of b. then $\lim_{x\to c} g(f(x)) = g(\lim_{x\to c} f(x))$

$\forall \varepsilon > 0$ s.t. $|g(f(x)) - g(b)| < \varepsilon$

g is continuous at b $\exists \delta_1 > 0$, s.t. if $|y-b| < \delta_1$, then $|g(y)-g(b)| < \varepsilon$

Since $\lim_{x\to c} f(x) = b$ $\exists \delta > 0$, $|f(x)-b| < \delta_1$ whenever $0<|x-0|<\delta$

• The Riemann's function

$\zeta(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}\ (\gcd(p,q)=1) \end{cases}$

Conclusion: ζ is continuous at all irrational numbers
discontinuous at all rational numbers

$x \in (0,1)$

e.g. $x = x_0 + \frac{1}{n}\pi$

① $x_0 = \frac{p}{q}$, take $\varepsilon = \frac{1}{2q}$. $\forall \delta > 0$, irrational numbers are dense in R. Choose $x \notin R$

$x \in (x_0-\delta, x_0+\delta)$, $|\zeta(x) - \zeta(x_0)| = \frac{1}{q_0} > \frac{1}{2q_0} = \varepsilon$. So $\zeta(x)$ is discontinuous at all rational numbers.

② x_0 is a irrational number $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $|x-x_0| < \delta$, $|\zeta(x) - \zeta(x_0)| = |\zeta(x)| < \varepsilon$

☆ finite sets

take $E(\varepsilon) = \{\frac{p}{q},\ 0 < \frac{1}{q} \le 1 \text{ and } q \le \frac{1}{\varepsilon}\}$, it is easy to show the number of $E(\varepsilon)$ is limited

$\hookrightarrow \frac{1}{q} \ge \varepsilon$

so $\forall x \in (x_0-\delta, x_0+\delta)$, we can choose x [a rational number but $x \notin E(\varepsilon)$]

let $\delta = \frac{1}{2}\mathrm{Min}\{|x_1-x_0|, |x_2-x_0|, \cdots |x_n-x_0|\}$ $(x_i \in E(\varepsilon))$ $\to \delta > 0$

so $|\zeta(x) - \zeta(x_0)| = |\frac{1}{q}| < \varepsilon$ so $\zeta(x)$ is continuous at all irrational numbers

$\hookrightarrow$ Select δ small enough s.t $(x_0-\delta, +x_0+\delta) \cap E(\varepsilon) = \emptyset$

• Heine $\lim_{x\to x_0} f(x)=A \Leftrightarrow \forall x_n\to x_0\ (n\to\infty),\ \lim_{n\to\infty} f(x_n)=A$

① $\forall \varepsilon>0, \exists \delta>0$ $|f(x)-A|<\varepsilon$ whenever $0<|x-x_0|<\delta$

$\exists N>0$, when $n>N$, $0<|x_n-x_0|<\delta$ we have $|f(x_n)-A|<\varepsilon$ so $\lim_{n\to\infty} f(x_n)=A$

② Assume $\lim_{x\to x_0} f(x)=A$ is not true

$\exists \varepsilon>0, \forall \delta>0$, $\exists x$ $0<|x-x_0|<\delta$, we have $|f(x)-A|\geq\varepsilon$ ①

take $\delta=\delta', \frac{\delta'}{2} \cdots, \frac{\delta'}{n}$ to the $\{x_n\}$, $0<|x_n-x_0|<\frac{\delta'}{n}$ but $|f(x_n)-A|\geq\varepsilon$;

we know that $\forall x_n\to x_0\ (n\to\infty)$ $\lim_{n\to\infty} f(x_n)=A \Rightarrow |f(x_n)-A|<\varepsilon$ ②

Since ① ② make contradictions, then we can conclude $\lim_{x\to x_0} f(x)=A$ is true

假设结论不成立 ⟹ 构造出一个数列 与已知条件(收敛)矛盾.

• $\mathbb{Q}$ is dense $\Leftrightarrow$ $a,b \xrightarrow{(a<b)}$ always can find c s.t $a<c<b$.

let $a,b\in\mathbb{Q}$ and $a<b$

we can find $n\in\mathbb{Z}$ s.t $\frac{1}{n}<b-a \Rightarrow a<b-\frac{1}{n}$

we also can find $m\in\mathbb{Z}$ s.t. $nb-1\leq m<nb$

then $b-\frac{1}{n}\leq\frac{m}{n}<b$

s.t $a<\frac{m}{n}<b$, hence

we can find $\frac{m}{n}\in\mathbb{Q}$ belongs to (a,b)

• Fermat Theorem

f has a local maximum or minimum value at an iterior point x_0 at its domain. $f'(x_0)$ is defined $\Rightarrow f'(x_0)=0$

x_0 is defined in (a,b), and have a derivative at $x=x_0$

if we have $f(x)\leq f(x_0)$ (or $f(x)\geq f(x_0)$). then $f'(x_0)=0$

let $f(x)\leq f(x_0)$

Pf: ① $x_0<x<x_0+\delta$ $f'(x_0)=f'_+(x_0)=\lim_{h\to0^+}\frac{f(x_0+h)-f(x_0)}{h}\leq0$

② $x_0-\delta<x<x_0$ $f'(x_0)=f'_-(x_0)=\lim_{h\to0^-}\frac{f(x_0+h)-f(x_0)}{h}\geq0$

then we can conclude $f'(x_0)=0$

• Rolle Theorem

$y=f(x)$ is continuous at $[a,b]$, differentiable at every point of its interior (a,b)

if $f(a)=f(b)$, there is at least exist one number $c\in(a,b)$ at which $f'(c)=0$

let $M \Rightarrow$ maximum, $m\Rightarrow$ minimum

① $M=m$ $\forall c\in(a,b)$. $f'(c)=0$

② $M>m$ just let $M\neq f(a)$, then we can find c s.t. $f(c)=M$

Given that $f(a)=f(b)$. $M>m$, S.t. M,m at least have one which $\neq f(a)$

so $\forall x\in[a,b]$, $f(x)\leq f(c)\Rightarrow f'(c)=0$

· $f(x)$ is continuous on $[0,1]$, $\int_0^1 f(x)\,dx = 0$

Show: ① $\exists \xi \in (0,1)$. $\int_0^{\xi} f(x)dx = f(\xi)$

② $\exists \eta \in (0,1)$. $\int_0^{\eta} f(x)\,dx = -\eta f(\eta)$

proof:

① let $F(x) = e^{-x} \int_0^x f(t)\,dt$

$F(0) = F(1) = 0 \Rightarrow \exists \xi \in (0,1)$ $F'(\xi) = -e^{-\xi}\int_0^{\xi} f(t)dt + e^{-\xi} f(\xi) = 0$

$\Rightarrow \int_0^{\xi} f(t)dt = f(\xi)$

② let $G(x) = x\int_0^x f(t)dt$

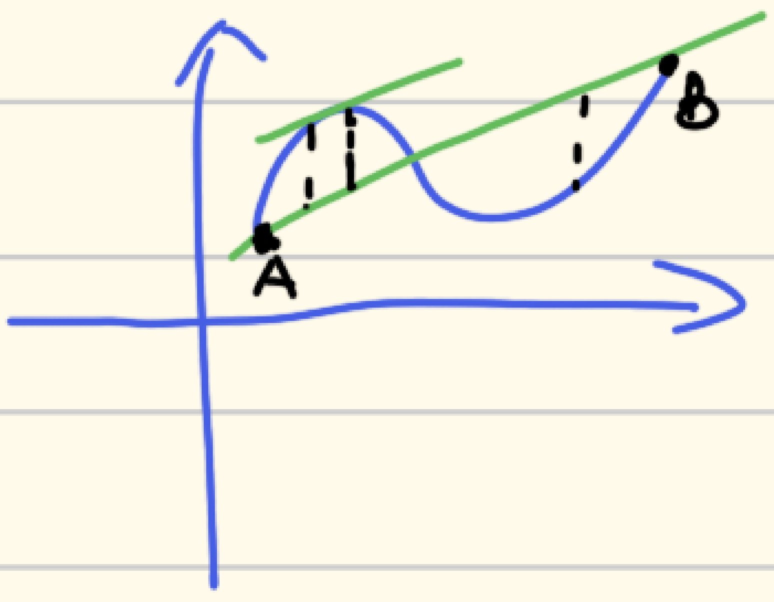
$G(0) = G(1) = 0 \Rightarrow \exists \eta \in (0,1)$ $F'(\eta) = \int_0^{\eta} f(t)dt + \eta f(\eta) = 0$

$\Rightarrow \int_0^{\eta} f(x)\,dx = -\eta f(\eta)$

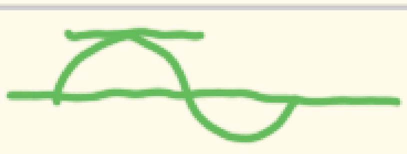
· Mean Value Theorem

$y=f(x)$ is continuous at $[a,b]$, differentiable at every point of its interior (a,b)

Then there is at least one point c in (a,b) at which $f'(c)=\frac{f(b)-f(a)}{b-a}$



rotate the function



$$\text{let}\quad \varphi(x)=f(x)-[\underbrace{f(a)+\frac{f(b)-f(a)}{b-a}(x-a)}_{y_{AB}}]$$

By Rolle Theorem, $\exists\, c\in[a,b]$

$$\varphi'(x)=f'(x)-\frac{f(b)-f(a)}{b-a},\qquad \varphi'(c)=0\Rightarrow \frac{f(b)-f(a)}{b-a}=f'(c)$$

导数

高阶导数：① $(u\cdot v)^{(n)}=\sum_{i=0}^{n} C_n^i\cdot u^{(i)}v^{(n-i)}$

② 利用泰勒公式 $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

$$=f(0)+f'(0)x+\cdots+\frac{f^{(n)}(0)}{n!}x^n$$

e.g. $f(x)=x^{22}e^{-3x^2}$. 求 $f^{(2022)}(0)=$___. $f(x)=x^{22}\sum_{i=0}^{\infty}\frac{(-3x^2)^i}{i!}=\sum_{i=0}^{\infty}\frac{(-3)^n x^{2n+22}}{n!}$ $\quad f^{(2022)}(0)\Rightarrow n=1000$

$$\Rightarrow f^{(2022)}(0)=\frac{2022!}{1000!}\cdot 3^{1000}$$

· 反函数的导数

① $\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}=\frac{1}{y'}$

② $\frac{d^2x}{dy^2}=\frac{d\frac{dx}{dy}}{dy}=\frac{\frac{1}{y'}}{dx}\cdot\frac{dx}{dy}=\frac{(y')^{-1}}{dx}\cdot\frac{1}{y'}$

$$=\frac{-(y')^{-2}\cdot y''}{y'}=-\frac{y''}{(y')^3}$$

数学常用的等价无穷小 $(x\to 0)$

① $\left.\begin{cases}\tan x\\ \sin x\\ \arcsin x\\ \arctan x\\ e^x-1\\ \ln(x+1)\end{cases}\right\}\to x$

② $\left.\begin{cases}1-\cos x\\ e^x-(x+1)\\ x-\ln(x+1)\end{cases}\right\}\to\frac{1}{2}x^2$

③ $\begin{cases}x-\sin x\sim\frac{1}{6}x^3\\ \tan x-x\sim\frac{1}{3}x^3\\ \tan x-\sin x\sim\frac{1}{2}x^3\end{cases}$ $\quad\begin{cases}x-\arcsin x\sim-\frac{1}{6}x^3\\ \arctan x-x\sim-\frac{1}{3}x^3\\ \arctan x-\arcsin x\sim-\frac{1}{2}x^3\end{cases}$

· 常用展开式

$\frac{1}{1-x}=1+x+x^2+\cdots x^n$

$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$

$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$

$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$

$\ln(x+1)=x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots$

$(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots$

$\tan x=1+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\cdots$

$x^a\sim a\ln x-1$ $(x\to 1)$

[e.g.1] $\lim_{x\to0}\frac{\tan\tan\tan x-\sin\sin\sin x}{\tan x-\sin x}$

$\square-\sin\square\sim\frac{1}{6}\square^3$
$\tan\square-\square\sim\frac{1}{3}\square^3$ → 等价条件 □ → 同样的一个函数

$$=\lim_{x\to0}\frac{\tan\tan\tan x-x+x-\sin\sin\sin x}{\frac{1}{2}x^3}$$

$$=2\lim_{x\to0}\frac{\tan\tan\tan x-\tan\tan x+\tan x\tan x-\tan x+\tan x-x+\boxed{}^{\text{类似处理}}}{x^3}$$

$$=2\cdot\left[\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)+\left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right)\right]=3$$

积分表:

$\int \tan x\,dx = -\ln|\cos x| + C$

$\int \cot x\,dx = \ln|\sin x| + C$

$\int \sec x\,dx = \ln|\sec x + \tan x| + C$

$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

$\int \frac{dx}{\cos^2 x} = \tan x + C$

$\int \frac{dx}{\sin^2 x} = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$

$\int \csc x \cot x\,dx = -\csc x + C$

$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$ $\quad(\cos^{-1}x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\tan^{-1}x)' = \frac{1}{x^2+1}$ $\quad(\sec^{-1}x)' = \frac{1}{|x|\sqrt{x^2-1}}$ $\quad(\csc^{-1}x)' = -\frac{1}{|x|\sqrt{x^2-1}}$

· 与三角相关的积分技巧

① $\sin^m x = \sin^{2k+1}x = (1-\cos^2 x)^k \cdot \sin x \Rightarrow -d(\cos x)$ $\quad$ m odd

② $\sin^m x \cos^n x = \sin^m x \cdot \cos^{2k+1} x = \sin^m x \cdot (1-\sin^2 x)^k \cdot \cos x \Rightarrow d(\sin x)$ $\quad$ m even $\quad$ n odd

③ $\sin^m x \cos^n x \Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$ $\quad \cos^2 x = \frac{1+\cos 2x}{2}$ $\quad$ 降次 $\quad$ m&n even

④ $\sec^2 x = \tan^2 x + 1$ $\quad$ e.g.

$$\begin{cases} \int \sec^3 x \tan^3 x\,dx = \int \sec^2 x(\sec^2 x - 1)\,d(\sec x) \\ \int \sec^3 x\,dx = \frac{1}{\cos^3 x}\,\frac{\cos x}{\cos x}\,dx = \frac{d(\sin x)}{(1-\sin^2 x)^2} \\ \int \sec^4 x \tan^3 x\,dx = (\tan^2 x + 1)\tan^3 x\,d(\tan x) \\ \int \sec^4 x \tan^2 x\,dx = (\tan^2 x + 1)\tan^2 x\,d(\tan x) \\ \int \tan^3 x\,dx = -\frac{1-\cos^2 x}{\cos^3 x}\,d(\cos x) \\ \int \tan^4 x\,dx = \int (\sec^2 x - 1)^2\,dx \end{cases}$$

⑤ 和差化积、

$$\begin{cases} \sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x] \\ \sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x] \\ \cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x] \end{cases}$$

⑥ $f(x)$为奇 $\int_{-a}^{a} f(x)\,dx = 0$，$f(x)$为偶 $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$\int_a^b \sqrt{r^2-x^2}\,dx \Rightarrow$ 考虑计算圆的面积

⑦ $f(x)$为连续函数

$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$

$\int_0^\pi f(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

$I = \int_0^\pi x f(\sin x)\,dx = \int_\pi^0 (\pi - t) f[\sin(\pi - t)](-dt)$

$= \int_0^\pi (\pi - x) f(\sin x)\,dx$ $\quad x + \pi - x = \pi$

$\Rightarrow 2\int_0^\pi x f(\sin x)\,dx = \pi \int_0^\pi f(\sin x)\,dx$

$\Rightarrow \int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$

⑧ $n \in N$ $\quad \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \begin{cases} \frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}, & n \text{ is even} \\ \frac{(n-1)!!}{n!!}, & n \text{ is odd} \end{cases}$

⑨ $\lim_{n\to\infty}\left(\frac{2n!!}{(2n-1)!!}\right)^2 \cdot \frac{1}{2n+1} = \frac{\pi}{2}$

⑩ let $t = \tan\frac{x}{2}$

$\Rightarrow \sin x = \frac{2t}{t^2+1}$, $\cos x = \frac{1-t^2}{t^2+1}$

$x = 2\arctan t \Rightarrow dx = \frac{2}{t^2+1}\,dt$

· 易错: $\lim_{x\to+\infty} \frac{e^x}{(1+\frac{1}{x})^{x^2}}$ $\Rightarrow$ 极限变量的趋向性是同时的，不能规定顺序 [极限与值存在区别]

错解: $\frac{e^x}{[(1+\frac{1}{x})^x]^x} = \frac{e^x}{e^x} = 1$

正解: $\frac{e^x}{e^{x^2\ln(1+\frac{1}{x})}} = e^{x - x^2\ln(\frac{1}{x}+1)} = e^{x - x^2(\frac{1}{x} - \frac{1}{2x^2} + o(\frac{1}{x^2}))} = e^{\frac{1}{2}+o(\frac{1}{x})} = e^{\frac{1}{2}}$

狄利克雷函数 & 黎曼函数

① 极限

(a) $\lim\limits_{x\to x_0} D(x)$ DNE

let $\varepsilon_0=\frac{1}{2}$ $\forall \delta>0$.

$\exists x_1, x_2 \begin{bmatrix} x_1 \text{ is a rational number} \\ x_2 \text{ is a irrational number}\end{bmatrix}$

$|D(x_1)-D(x_2)|=1>\varepsilon_0$

(b) $\forall \varepsilon>0 \Rightarrow \underbrace{n\le\frac{1}{\varepsilon}}$ n is a finite number

let $\delta>0$ s.t $U^\circ(x_0;\delta)$ exclusive $\{\frac{p}{q}\}$

$|R(x)-0|=|R(x)|<\varepsilon$ $\qquad R(\frac{p}{q})=\frac{1}{q}\ge\varepsilon$

$\lim\limits_{x\to x_0} R(x)=0$

② 连续性

(a) 均不连续

(b) ① x_0 is a rational number $\qquad$ irr

$R(x_0)=\frac{1}{q}$ $\qquad$ 稠密 $\Rightarrow |R(x_n)-R(x_0)|=\frac{1}{q}>0$

② x_0 is a irrational number

同有限个 $\frac{p}{q}$ 思路

黎曼可积

1. 定义 $P=\{x_0,\cdots,x_n\}$ $[a,b]$

$\xi_i\in[x_{i-1},x_i]$ $\quad \lambda=\max\limits_{1\le i\le n}(\Delta x_i)$

$\lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i$ is exists

与 P, ξ_i 无关

2. 必要条件 可积 → 有界

Assume $f(x)$ is unbounded in $[a,b]$

$G=\left|\sum\limits_{i\ne k} f(\xi_i)\Delta x_i\right|$

Since $f(x)$ is unbounded

$\forall M>0$, $\exists \xi_k\in\Delta k$ $|f(\xi_k)|>\frac{M+G}{\Delta x_k}$

$\left|\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i\right| \ge |f(\xi_k)\Delta x_k|-\left|\sum\limits_{i\ne k} f(\xi_i)\Delta x_i\right|$

$=\frac{M+G}{\Delta x_k}\cdot\Delta x_k-G=M$

3. 充要条件

$\begin{cases} M_i=\sup\limits_{x\in\Delta i} f(x) \\ m_i=\inf\limits_{x\in\Delta i} f(x)\end{cases}$

达布和

$m(b-a)\le s(f,P)\le\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i\le S(f,P)\le M(b-a)$

$\forall\varepsilon>0$, $\exists\delta>0$

$\|P\|<\delta \Rightarrow \sum\limits_{i=1}^{n}\omega_i\Delta x_i<\varepsilon$

$\forall\varepsilon>0$. $\exists P=\{x_0,\cdots,x_n\}$

$S(f,P)-s(f,P)<\varepsilon$

4. 充分条件

$\begin{cases} [a,b] \text{上连续} \\ [a,b] \text{上只有有限个间断点} \Rightarrow \text{可积} \\ [a,b] \text{上单调}\end{cases}$

examine the intergrability of a function

<1> $\quad f(x)=\begin{cases} x & x \text{ is rational} \\ -x & x \text{ is irrational}\end{cases}$

Assume $f(x)$ is intergrable.

$P=\{x_0,x_1,\cdots x_n\}$ $\quad 0\le x_0\le x_1\le\cdots\le x_{n-1}\le x_n=1$

$\begin{cases} M_i=\sup\limits_{x_{i-1}\le x\le x_i} f(x)=x_i \text{ (rational)} \\ m_i=\inf\limits_{x_{i-1}\le x\le x_i} f(x)=-x_{i-1} \text{ (irrational)}\end{cases}$

$S(f,P)-s(f,P)=\sum\limits_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})$

$=x_n^2-x_0^2=1>\varepsilon \quad (\text{let }\varepsilon=\frac{1}{2})$

<2>

Question 3 (15 marks). Let f be a monotone function defined on $\mathbb{R}=(-\infty,\infty)$. continuous

(a) Prove that f is integrable over any bounded and closed interval.

(b) Can you conclude that f has an antiderivative on $\mathbb{R}$? If your answer is "YES" then give a proof; if your answer is "NO" then give a counterexample.

(a) $f(a)\le f(x)\le f(b)$ $\quad x\in[a,b]$

$M=\max\{|f(a)|,|f(b)|\}$

$P=\{x_0,\cdots\cdots,x_n\}$, $\varepsilon>0$ s.t. $\|P\|<\frac{\varepsilon}{2M}$

$\Delta(f,[a,b],P)=\sum\limits_{k=1}^{n}\omega(f,I_k)\Delta x_k$

$=\sum\limits_{k=1}^{n}[f(x_k)-f(x_{k-1})]\Delta x_k$

$\le\|P\|\sum\limits_{k=1}^{n}[f(x_k)-f(x_{k-1})]\le\|P\|(f(b)-f(a))\le 2M\|P\|<\varepsilon$

Question 4 (20 marks). Examine the integrability of the given function on the interval $[0,1]$, and prove your conclusion.

(a) $f(x) = \begin{cases} x^2, & \text{if } x \text{ is a rational number,} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$

(b) $g(x) = \begin{cases} \sin\left(\dfrac{1}{\sin\frac{1}{x}}\right), & \text{if } x \neq 0 \text{ and } \frac{1}{\pi x} \text{ is not an integer} \\ 1, & \text{if } x = 0, \text{ or if } \frac{1}{\pi x} \text{ is an integer.} \end{cases}$

(a) $P_n = \{x_0, \cdots x_n\}$ $\quad c_k = \frac{k}{n}$ (rational)

$$S(f, P_n, \xi) = \sum_{i=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \to \frac{1}{3} \text{ as } n \to \infty$$

$c_k' \in \left(\frac{k-1}{n}, \frac{k}{n}\right)$. $S(f, P_n, \xi') = \sum_{k=1}^{n} f'(c_k')\Delta x = 0$

$\exists N_1 > 0$, $S(f, P_n, \xi) > \frac{1}{4}$. $\forall n > N_1$

$\forall \delta > 0$, take $N > \max\{N_1, \frac{1}{\delta}\}$; then $n > N$.

we have $\|P_n\| = \frac{1}{n} < \frac{1}{N} < \delta$

$$S(f, P_n) - s(f, P_n) \geq S(f, P_n, \xi) - S(f, P, \xi') > \frac{1}{4}$$

- $f(x) = \begin{cases} x & \text{rational} \\ 2x & \text{irrational} \end{cases}$

① rational $\quad I_k = \left[\frac{k-1}{n}, \frac{k}{n}\right]$, $C_k = \frac{k-1}{n}$

$$s(f, P_n, \xi) = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} \frac{1}{n^2}\frac{n(n-1)}{2} \to \frac{1}{2} \text{ as } n \to \infty$$
$$\leq \frac{5}{8}$$

② irrational $\quad C_k' \in \left(\frac{k-\frac{1}{2}}{n}, \frac{k}{n}\right)$

$$f(C_k') = 2C_k' \geq \frac{2k-1}{n}$$

$$S(f, P_n, \xi) \geq \sum_{k=1}^{n} = \frac{1}{n^2}\sum_{k=1}^{n}(2k-1) \to 1 \quad \left[\frac{7}{8}\right] \text{ as } n \to \infty$$

$$S(f, P_n, \xi) - s(f, P_n, \xi) \geq \frac{7}{8} - \frac{5}{8} = \frac{1}{4} \quad \text{as } \lim_{\|P\| \to 0} \Delta(P) > 0$$

- $f(x) = \begin{cases} 1 - \frac{1}{n} & x \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right), n = 1, 2 \cdots \\ 1 & x = 0 \end{cases}$

$\forall \varepsilon > 0$, $\quad \exists m > 0$ s.t. $\frac{1}{2^m} < \frac{\varepsilon}{2}$

in $\left[\frac{1}{2^m}, 1\right]$, f is bdd & has finite discontinuous points $\Rightarrow$ f is integrable in $\left[\frac{1}{2^m}, 1\right]$

$\exists \varepsilon > 0$ s.t. $\sum_{i=1}^{n} w_i \Delta x_i' < \frac{\varepsilon}{2}$ in $\left[0, \frac{1}{2^m}\right]$

$$\Rightarrow \sum_{i=1}^{n} w_i \Delta x_i^2 + \sum_{i=1}^{n} w_i \Delta x_i' < \varepsilon$$

- $f(x)$ is bounded on $[a,b]$, $\{a_n\} \subseteq [a,b]$, $\lim_{n\to\infty} a_n = \varepsilon$.

if $f(x)$ have finite discontinuity point $\{a_1, a_2, \cdots a_n\}$ on $[a,b]$

show $f(x)$ is integrable on $[a,b]$

Proof: a ① $a+\delta$ ② b

let $\varepsilon = a$.

$f(x)$ is bounded on $[a,b] \Rightarrow \exists M > 0$. s.t $|f(x)| \leq M$ then $W_i = \sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x) \leq 2M$

$\forall \varepsilon > 0$. $\delta = \min\{\frac{\varepsilon}{4M}, \frac{b-a}{2}\}$ $\qquad$ Since $\lim_{n\to\infty} a_n = a$, in ② interval has finite discontinuity point,

$\Rightarrow f(x)$ is integrable on ②, $\exists [a+\delta, b]$ partition $P' = \{\Delta_2, \Delta_3 \cdots \Delta_n\}$, let $\Delta_1 = [a, a+\delta]$

s.t. $P = \{\Delta_1, \Delta_2, \cdots \Delta_n\}$ $\qquad \sum_P W_i \Delta x_i = W_1 x_1 + \sum_{P'} W_i \Delta x_i \leq 2M\delta + \frac{\varepsilon}{2} = \varepsilon$

(given that $a + \delta < b \Rightarrow \delta < b - a$)

• $f(x)$ is integrable on $[a,b]$ and $|f(x)| \geq m > 0$, show $\frac{1}{f(x)}$ is integrable on $[a,b]$

Proof:

$\forall \varepsilon > 0$, $\exists P$ s.t. $\sum_P \omega_i^f \Delta x_i < \varepsilon$

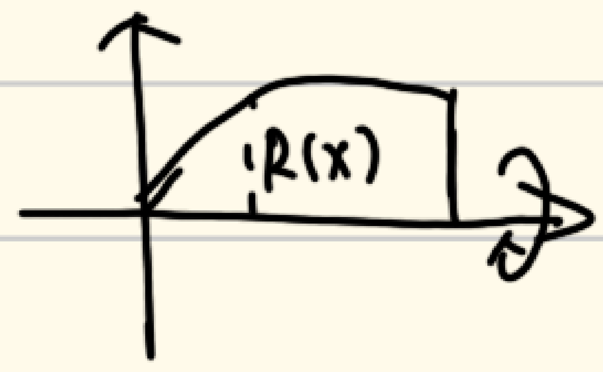
$\exists x', x''$ $[|f(x)| \geq m > 0]$

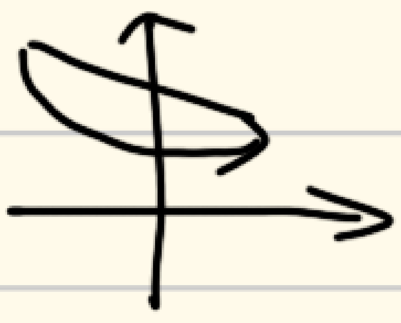
$$\left|\frac{1}{f(x')} - \frac{1}{f(x'')}\right| = \left|\frac{f(x'') - f(x')}{f(x')f(x'')}\right| \leq \frac{\omega_i^f}{m^2} \quad \text{i.e. } \omega_i^{\frac{1}{f}} \leq \frac{\omega_i^f}{m^2}$$

$$\sum_P \omega_i^{\frac{1}{f}} \Delta x_i \leq \frac{1}{m^2} \sum_P \omega_i^f \Delta x_i < \frac{\varepsilon}{m^2} \Rightarrow \frac{1}{f(x)} \text{ is integrable on } [a,b]$$

定积分应用

① $V=\int_a^b \underset{\text{area}}{A(x)}\,dx$

②  $V=\int_a^b \pi[R(x)^2]\,dx$

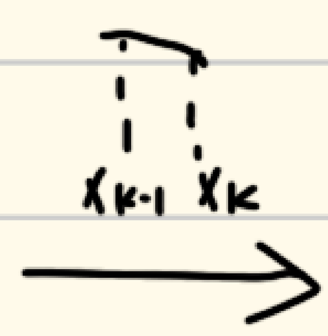
③  $V=\int_a^b \pi\left([R(x)]^2-[r(x)]^2\right)dx$

④ 

$V=\sum_{k=1}^{n}\Delta V_k=\sum_{k=1}^{n} h\,\Delta x_k\,2\pi(R_k-r_k)=\int_a^b 2\pi(x-L)f(x)\,dx$

⑤ the length of the curve $\Delta L=\sqrt{\Delta x^2+\Delta y^2}=\sqrt{1+f'(x)^2}\,\Delta x$

$\Rightarrow L=\int_a^b\sqrt{1+f'(x)^2}\,dx$

⑥ Surface area  $\Delta S=2\pi R\,\Delta L=2\pi\cdot\frac{f(x_{k-1})+f(x_k)}{2}\cdot\sqrt{\Delta x_k^2+\Delta y_k^2}$

$\Rightarrow S=\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$

积分第一中值定理:

f is continuous on $[a,b]$ $\exists\xi\in(a,b)$, $\int_a^b f(x)\,dx=f(\xi)(b-a)$

Proof: $m\le f(x)\le M\Rightarrow m(b-a)\le\int_a^b f(x)\,dx\le M(b-a)\Rightarrow f(\xi)=\frac{\int_a^b f(x)}{b-a}$ $\xi\in[a,b]$

推广 $f.g$ is continuous on $[a,b]$. $g(x)$ 在$[a,b]$不变号. $\exists\xi\in[a,b]$ $\int_a^b f(x)g(x)\,dx=f(\xi)\int_a^b g(x)\,dx$

Proof: let $g(x)\ge0$. then $mg(x)\le f(x)g(x)\le Mg(x)\Rightarrow m\int_a^b g(x)\,dx\le\int_a^b f(x)g(x)\,dx\le M\int_a^b g(x)\,dx$

① $\int_a^b g(x)\,dx=0\Rightarrow\int_a^b f(x)g(x)\,dx=0$

② $\int_a^b g(x)\,dx>0$ $m\le\frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}\le M$

$\Rightarrow\exists\xi\in[a,b]$ $f(\xi)=\frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}$

积分第二中值定理:

f is integrable on $[a,b]$ and $g(x)$ is monotone on $[a,b]$

$\int_a^b f(x)g(x)\,dx=g(a)\int_a^\xi f(x)\,dx+g(b)\int_\xi^b f(x)\,dx$

$=F(x)g(x)\Big|_a^b-\int_a^b F(x)\,d(g(x))$ $\left(F(x)=\int_a^x f(t)\,dt\right)$ $F(a)=0$

$=F(b)g(b)-F(a)g(a)-\int_a^b F(x)g'(x)\,dx$

$=F(b)g(b)-F(a)g(a)-F(\xi)\int_a^b g'(x)\,dx$ ($g'(x)$不变号)

$=F(b)g(b)-F(a)g(a)-F(\xi)(g(b)-g(a))$

$=g(b)\int_a^b f(x)\,dx-[g(b)-g(a)]\int_a^\xi f(x)\,dx$

$=g(b)\left[\int_a^b f(x)\,dx-\int_a^\xi f(x)\,dx\right]+g(a)\int_a^\xi f(x)\,dx$

$=g(b)\int_\xi^b f(x)\,dx+g(a)\int_a^\xi f(x)\,dx$

$\Rightarrow$ [e.g.1] $\int_a^b f(x)g(x)\,dx=f(a)\int_a^\xi g(x)\,dx$ [$g(x)$ is decreasing] $=g(b)\int_\xi^b f(x)\,dx$ [$g(x)$ is increasing]

[e.g.2] $\left|\int_x^{x+1}\sin(t^2)\,dt\right|\le\frac{1}{x}$

let $u=t^2$, $F(x)=\int_x^{x+1}\sin(t^2)\,dt$

$\Rightarrow |F(x)|=\left|\frac{1}{2}\int_{x^2}^{(x+1)^2}\frac{\sin u}{\sqrt u}\,du\right|$

$f(x)=\sin u$, $g(u)=\frac{1}{\sqrt u}$ $|F(x)|=\left|\frac{1}{2}\frac{1}{\sqrt{x^2}}\int_{x^2}^{\xi}\sin u\,du\right|=\frac{1}{2x}\left|\int_{x^2}^{\xi}\sin u\,du\right|$

$=\frac{1}{2x}\left|\cos x^2-\cos\xi\right|\le\frac{1}{2x}\left(|\cos x^2|+|\cos\xi|\right)\le\frac{1}{2x}(1+1)=\frac{1}{2x}\cdot2=\frac{1}{x}$

反常积分

1. Cauchy Criterion

$\lim_{x\to c} f(x)$ exists $\Leftrightarrow \forall \varepsilon>0, \exists\, 0<\delta<\delta_0$ s.t. $|f(x)-f(x')|<\varepsilon \quad x, x' \in (c-\delta, c)\cup(c, c+\delta)$

$\lim_{x\to\infty} f(x)$ exists $\Leftrightarrow \forall \varepsilon>0, \exists M>a$. s.t. $|f(x)-f(x')|<\varepsilon \quad x, x'>M$

2. Direct Comparison Test

$0\le f(x)\le g(x)$ as $x\ge a$ $\begin{cases} \int_a^\infty g(x)dx \text{ converges} \Rightarrow \int_a^\infty f(x)dx \text{ converges} \\ \int_a^\infty f(x)dx \text{ diverges} \Rightarrow \int_a^\infty g(x)dx \text{ diverges} \end{cases}$

· Limit Comparison Test [$f(x), g(x)$ positive functions]

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \quad (0<L<\infty)$ $\quad f(x), g(x)$ both converge/diverge

3. Dirichlet's test [变号]

$F(u)=\int_a^u f(x)dx$ is bounded on $[a,+\infty)$ $\quad g(x)$ is monotone on $[a,+\infty)$, $\lim_{x\to\infty} g(x)=0$

then $\int_a^\infty f(x)g(x)dx$ converges

Proof: let $|F(A)|\le C \quad \forall A\ge a$, then

$\left|\int_A^B f(x)dx\right| = \left|\int_a^B f(x)dx - \int_a^A f(x)dx\right| \le 2C \quad \forall A, B\ge a$

Since $\lim_{x\to\infty} g(x)=0 \quad \forall\varepsilon>0, \exists M>0$, when $x>M$, $|g(x)|\le\frac{\varepsilon}{4C}$

$$\left|\int_A^B f(x)g(x)dx\right| = \left|g(A)\int_A^\xi f(x)dx + g(B)\int_\xi^B f(x)dx\right|$$
$$\le \frac{\varepsilon}{4C}\left|\int_A^\xi f(x)dx\right| + \frac{\varepsilon}{4C}\left|\int_\xi^B f(x)dx\right|$$
$$\le \frac{\varepsilon}{4C}\cdot 2C + \frac{\varepsilon}{4C}\cdot 2C = \varepsilon$$

4. Abel

$\int_a^\infty f(x)dx$ is bounded, $g(x)$ is monotone and bounded $\Rightarrow \int_a^\infty f(x)g(x)dx$ converges

$|g(x)|\le C \quad \forall x\in[a,+\infty)$, $\forall\varepsilon>0, \exists M$. $\left|\int_A^B f(x)dx\right|\le\frac{\varepsilon}{2C}$ when $A, B>M$

$\left|\int_A^B f(x)g(x)dx\right| = \left|g(A)\int_A^\xi f(x)dx + g(B)\int_\xi^B f(x)dx\right| \le C\left|\int_A^\xi f(x)dx\right| + C\left|\int_\xi^B f(x)dx\right| \le \varepsilon$

· $\int_1^\infty \frac{1}{x^p\ln^q x} = \int_1^2 \frac{1}{x^p \ln^q x} + \int_2^\infty \frac{1}{x^p\ln^q x}$ $\quad$ $\to p>1$ and $q<1 \Rightarrow$ converges $\quad$ [$\ln x$ 与 $x-1$ 等价无穷小 $(x\to1)$]

① $x\to1^+$

$$\lim_{x\to1^+}\frac{\frac{1}{x^p\ln^q x}}{\frac{1}{(x-1)^q}} = \lim_{x\to1^+}\frac{(x-1)^q}{\ln^q x} = 1 \Rightarrow \begin{array}{l} q<1 \text{ converges} \\ q\ge1 \text{ diverges}\end{array}$$

② $x\to\infty$ $\quad$ when $q<1$

if $p>1$ $\quad\exists\, 1<p_1<p$

$$\lim_{x\to\infty}\frac{\frac{1}{x^p\ln^q x}}{\frac{1}{x^{p_1}}} = \lim_{x\to\infty}\frac{1}{x^{p-p_1}\ln^q x} = 0$$

converges

if $p<1$, $\exists\, p<p_2<1$

$$\lim_{x\to\infty}\frac{\frac{1}{x^p\ln^q x}}{\frac{1}{x^{p_2}}} = \lim_{x\to\infty}\frac{x^{p_2-p}}{\ln^q x} = \infty$$

diverges

if $p=1$, $\int_2^\infty \frac{dx}{x\ln^q x} = \int_{\ln 2}^\infty \frac{dy}{y^q}$

diverges

· $\int_1^\infty \frac{\sin x}{x^p}dx$

① $p>1 \quad \int_1^\infty \frac{\sin x}{x^p}dx \le \int_1^\infty\frac{|\sin x|}{x^p}dx \le \int_1^\infty \frac{1}{x^p}dx$ 绝对收敛

② $0<p<1 \quad \frac{1}{x^p}$ 在 $[1,+\infty)\downarrow$ 且 $\lim_{x\to\infty}\frac{1}{x^p}=0$, $\left|\int_1^\infty \sin x\right|\le 2$

对于 $\int_1^\infty \frac{|\sin x|}{x^p}dx > \frac{|\sin x|}{x} \Rightarrow$ 条件收敛

③ $p=1$ 由 Dirichlet 判断法知收敛

$\int_1^\infty \frac{|\sin x|}{x} \ge \int_1^\infty \frac{\sin^2 x}{x} = \int_1^\infty \frac{1-\cos 2x}{2x}dx = \int_1^\infty\left(\frac{1}{2x} - \frac{\cos 2x}{2x}\right)dx$ ($\frac{1}{2x}$: 发散)

$\Rightarrow$ 条件收敛

· $\int_1^\infty \ln\left(\cos\frac{1}{x}+\sin^p\frac{1}{x}\right)dx$

when $x\to\infty \Rightarrow \int_1^\infty \ln\left(1-\frac{1}{2x^2}+\frac{1}{x^p}+o\left(\frac{1}{x^2}+\frac{1}{x^p}\right)\right)dx = \int_1^\infty \left(\underbrace{-\frac{1}{2x^2}}_{I_1} + \underbrace{\frac{1}{x^p}}_{I_2} + \underbrace{o\left(\frac{1}{x^q}\right)}_{I_3}\right)dx$ $\quad I_1$ 收敛

$q=\min\{2,p\}$

$I_2 \begin{cases} 0<p\le1 & \text{发散} \quad \Rightarrow \text{发散} \\ p>1 & \text{收敛} \quad q=\min\{2,p\}>1 \Rightarrow \text{收敛} \end{cases}$

• $\int_0^{\infty} \frac{\sin bx}{x^{\lambda}} dx \quad (b \neq 0)$

let $t = bx$

$= b^{\lambda-1} \int_0^{\infty} \frac{\sin t}{t^{\lambda}} dt$

① $\int_0^1 \frac{\sin x}{x^{\lambda}} dx$

1° $\lambda \leq 0$，为正常积分 且 $\frac{\sin x}{x^{\lambda}} > 0$. abs. conv.

2° $\lambda > 0$

$\frac{\sin x}{x^{\lambda}} \sim \frac{1}{x^{\lambda-1}}$ as $x \to 0^+$

$\lim_{x \to 0^+} x^{\lambda-1} \cdot \frac{\sin x}{x^{\lambda}} = 1 \begin{cases} \lambda - 1 < 1, \text{ i.e. } \lambda < 2 \text{ abs. conv.} \\ \lambda \geq 2. \text{ div.} \end{cases}$

② $\int_1^{\infty} \frac{\sin x}{x^{\lambda}} dx$

1° $\lambda > 1$

since $\left|\frac{\sin x}{x^{\lambda}}\right| \leq \frac{1}{x^{\lambda}}$. when $\lambda > 1$. abs. conv.

2° $0 < \lambda \leq 1$

$\begin{cases} \left|\int_1^A \sin x\, dx\right| \text{ is bdd.} \\ \frac{1}{x^{\lambda}} = 0 \text{ as } x \to \infty \end{cases} \Rightarrow$ conv.

$\left|\frac{\sin x}{x^{\lambda}}\right| \geq \frac{\sin^2 x}{x^{\lambda}} = \frac{1}{2} \cdot \frac{1-\cos 2x}{x^{\lambda}}$ div.

$\Rightarrow$ con. conv.

3° $\lambda \leq 0$ • $\lambda = 0$ $\frac{\sin x}{x^{\lambda}} = \sin x$ div

• $\lambda < 0$ $x \in [2n\pi + \frac{\pi}{4}, 2n\pi + \frac{3}{4}\pi]$ $\sin x \geq \frac{\sqrt{2}}{2}$ $\frac{1}{x^{\lambda}} \geq 1$

$\left|\int_{2n\pi+\frac{\pi}{4}}^{2n\pi+\frac{3}{4}\pi} \frac{\sin x}{x^{\lambda}} dx\right| \geq \int 1 \cdot \frac{\sqrt{2}}{2} \cdot (\frac{3}{4}\pi - \frac{\pi}{4}) dx = \int \frac{\sqrt{2}}{4}\pi\, dx$ div.

Above all $\begin{cases} \lambda \in (0,1] \text{ con. conv.} \\ \lambda \in (1,2) \text{ abs. conv.} \\ \text{otherwise div.} \end{cases}$

常微分方程的通解

$\frac{dy}{dx} + P(x)y = Q(x)$

$u(x)\frac{dy}{dx} + p(x)u(x)y = Q(x)u(x)$

$(\square \cdot y)' = Q(x)u(x)$

$u(x)p(x) = u'(x)$ [找一组解]

$\frac{du}{dx} = u \cdot p(x)$

$\ln|u| = \int p(x)dx \boxed{+C}$ 没必要，找出特解即可

$u = e^{\int p(x)dx}$

$(e^{\int p(x)dx} \cdot y)' = Q(x) \cdot e^{\int p(x)dx}$

$e^{\int p(x)dx} \cdot y = \int Q(x) e^{\int p(x)dx} dx$

$y^* = e^{-\int p(x)dx} \int Q(x) \cdot e^{\int p(x)dx} dx$ 一组特解

$y' + p(x)y = 0$ 通解: $y = C \cdot e^{-\int p(x)dx}$

$\Rightarrow y' + p(x)y = Q(x)$ 通解

$y = e^{-\int p(x)dx}\left(\int Q(x) \cdot e^{\int p(x)dx} dx + C\right)$

MAT 1011 4次quiz 压轴

· Quiz 1

f is continuous function defined on $[-1,1]$ and assumed $\lim_{x\to 0} f(\sin\frac{1}{x})$ exists. Show $f(x)$ is identically constant on $[-1,1]$.

proof: Suppose not. $\exists a\in[-1,1]$, $f(a)\neq f(0)$

Since $a\in[-1,1]$ $\exists\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$ s.t. $\sin\theta=a$

let $x_n=\frac{1}{\theta+2n\pi}$ $\Rightarrow f(\sin\frac{1}{x_n})=f(a)$

let $y_n=\frac{1}{2n\pi}$ $\Rightarrow f(\sin\frac{1}{y_n})=f(0)$

as $n\to\infty$ $x_n\to 0$, $y_n\to 0$, but $\lim_{x_n\to 0} f(\sin\frac{1}{x_n})=\lim_{y_n\to 0} f(\sin\frac{1}{y_n})$

which contradicts to $f(a)\neq f(0)$

s.t. $f(x)$ is identically constant on $[-1,1]$

· Quiz 2

$f(x)$ is defined on $(-2,2)$, $H(x)$ is defined on I and differentiable at $x=0$

G is defined on $\mathbb{R}$ and differentiable at $y_0=f(0)$. Assume:

(a) $\lim_{n\to\infty} n\{f(\frac{1}{n})-f(0)\}=L$

(b) for all integer $n\geq 1$, $G(f(\frac{1}{n}))=H(\frac{1}{n})f(\frac{1}{n})$

(c) $G'(y_0)\neq H(0)$

find L and prove your result

Proof:

by (a): $\lim_{n\to\infty}\frac{f(\frac{1}{n})-f(0)}{\frac{1}{n}}=L$

$$f(\tfrac{1}{n})-f(0)=\frac{L+\xi_1(n)}{n}\quad(\lim_{n\to\infty}\xi_1(n)=0)\quad ①$$

Since $H'(0)$ exists, then

$H(x)=H(0)+(H'(0)+\xi_2(x))x$ $(\lim_{x\to 0}\xi_2(x)=0)$

$$\Rightarrow H(\tfrac{1}{n})=H(0)+\frac{H'(0)+\xi_3(n)}{n}\quad(\lim_{n\to\infty}\xi_3(n)=0)\quad ②$$

by ①②:

$$H(\tfrac{1}{n})f(\tfrac{1}{n})=H(0)f(0)+\frac{H'(0)f(0)+H(0)L+\xi_4(n)}{n}$$

$$\xi_4(n)=H(0)\xi_1(n)+\frac{H'(0)(L+\xi_1(n))}{n}+f(0)\xi_3(n)+\frac{(L+\xi_1(n))\xi_3(n)}{n}\quad(\lim_{n\to\infty}\xi_4(n)=0)\quad ③$$

Since $G'(y_0)$ exists, $G(y)=G(y_0)+(G'(y_0)+\eta_1(\Delta y))\Delta y$ $(\lim_{\Delta y\to 0}\eta_1(\Delta y)=0)$

take $y=y_n=f(\frac{1}{n})$, $y_0=f(0)$ $\Delta y=f(\frac{1}{n})-f(0)$, $\eta_2=\eta(\frac{1}{n})$

$\Delta y=\frac{L+\xi_1(n)}{n}\to 0$, $\eta_2(n)\to 0$ as $n\to\infty$

$$G(f(\tfrac{1}{n}))=G(y_0)+(G'(y_0)+\eta_2(n))\frac{L+\xi_1(n)}{n}\quad ④$$

plugging ③④ into (b)

$$G(y_0)+(G'(y_0)+\eta_2(n))\frac{L+\xi_1(n)}{n}=H(0)f(0)+\frac{H'(0)f(0)+H(0)L+\xi_4(n)}{n}$$

Since, $G(y_0)=H(0)f(0)$ as $n\to\infty$

$$G'(y_0)L=H'(0)f(0)+H(0)L$$

$$\Rightarrow L=\frac{H'(0)f(0)}{G'(y_0)-H(0)}$$

• Quiz 3

Assume $f(x)$ is continuously differentiable on a $[a,b]$. If P is a partition of $[a,b]$ and $S(f,P)$ is a Riemann sum of f associated with the partition P. Show:

$$\left| S(f,P) - \int_a^b f(x)\,dx \right| \le \|P\| \int_a^b |f'(x)|\,dx$$

Proof: $P: a \le x_0 < x_1 < \cdots < x_n = b$

$$f(b_k) = \max_{[x_{k-1},x_k]} f(x) \qquad f(a_k) = \min_{[x_{k-1},x_k]} f(x)$$

$$S(f,P) - \int_a^b f(x)\,dx = \sum_{k=1}^{n} \left[f(c_k)\Delta x_k - \int_{x_{k-1}}^{x_k} f(x)\,dx \right]$$

$$= \sum_{k=1}^{n} \left(f(c_k) - f(\xi_k) \right)\Delta x_k \le \sum_{k=1}^{n} [f(b_k) - f(a_k)]\Delta x_k$$

$$= \sum_{k=1}^{n} \left| \int_{a_k}^{b_k} f'(x)\,dx \right| \Delta x_k \le \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f'(x)|\,dx\, \|P\|$$

$$= \|P\| \int_a^b |f'(x)|\,dx$$

• Quiz 4

for any constant $a > 0$ and any positive integer n let $C(n,a)$ be a constant defined by

$$C(n,a) = \frac{(-1)^n}{a^{2n+1}} \left\{ \int_0^a \frac{\ln(x^2+1)}{x^2}\,dx - \sum_{k=1}^{n} (-1)^{k-1} \frac{a^{2k-1}}{k(2k-1)} \right\}$$

Prove that

$$\frac{1}{(n+1)(2n+1)(1+a^2)^{n+1}} \le C(n,a) \le \frac{1}{(n+1)(2n+1)}$$

$$\ln(y+1) = \sum_{k=1}^{n} (-1)^{k-1} \cdot \frac{y^k}{k} + (-1)^n \frac{y^{n+1}}{(n+1)(1+\theta y)^{n+1}} \quad \theta \in (0,1)$$

$$\frac{\ln(x^2+1)}{x^2} = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-2}}{k} + (-1)^n f_n(x) \qquad \left(f_n(x) = \frac{x^{2n}}{(n+1)(1+\theta x^2)^{n+1}} \right)$$

Since $1 \le 1+\theta x^2 \le 1+a^2$, $\dfrac{x^{2n}}{(n+1)(1+a^2)^{n+1}} \le f_n(x) \le \dfrac{x^{2n}}{n+1}$

$$\int_0^a \frac{x^{2n}}{(n+1)(1+a^2)^{n+1}}\,dx \le \int_0^a f_n(x)\,dx \le \int_0^a \frac{x^{2n}}{n+1}\,dx \quad ①$$

$$\Rightarrow \frac{a^{2n+1}}{(n+1)(2n+1)(1+a^2)^{n+1}} \le \int_0^a f_n(x)\,dx \le \frac{a^{2n+1}}{(n+1)(2n+1)}$$

$$C(n,a) = \frac{(-1)^n}{a^{2n+1}} \int_0^a \left(\frac{\ln(x^2+1)}{x^2} - \sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-2}}{k} \right) dx$$

$$= \frac{(-1)^n}{a^{2n+1}} \int_0^a f_n(x)\,dx \quad ②$$

by ① ②

$$\Rightarrow \frac{1}{(n+1)(2n+1)(1+a^2)^{n+1}} \le C(n,a) \le \frac{1}{(n+1)(2n+1)}$$