• Vector

$<\vec{v},\vec{w}> = \sum_{i=1}^{n} v_i w_i$

• Matrix - Vector product

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ A m×n matrix

$A\vec{u} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} u_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} u_j \end{bmatrix}$ m×1

• colmun-form

A m×n $\vec{w}$ n×1

$A = [\vec{a_1}, \vec{a_2} \cdots \vec{a_n}]$

$A\vec{w} = w_1\vec{a_1} + w_2\vec{a_2} + \cdots + w_n\vec{a_n}$

三种表示: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$A\vec{x} =$

① 按行点积 $A\vec{x} = \begin{pmatrix} (1,2)\cdot\binom{5}{6} \\ (3,4)\cdot\binom{5}{6} \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$

② 线性组合 系数×基向量 $A\vec{x} = 5\cdot\binom{1}{3} + 6\binom{2}{4} = \binom{17}{39}$

③ 矩阵

• Matrix

• lower/upper triangular matrix

左下 右上

Diagonal matrix $\forall i \neq j$ $D_{i,j} = 0$

只有对角线上的元素可能不为0

1. prove $(AB)C = A(BC)$

A: m×n B: n×r C: r×s

$D = AB$, $E = BC$

⇒ show $DC = AE$

$DC = \sum_{k=1}^{r} d_{ik} c_{kj} = \sum_{k=1}^{r} \left(\sum_{l=1}^{n} a_{il} b_{lk}\right) c_{kj}$

$AE = \sum_{k=1}^{n} a_{ik} \cdot e_{kj} = \sum_{k=1}^{n} a_{ik}\left(\sum_{l=1}^{r} b_{kl} \cdot c_{lj}\right)$

$\Rightarrow DC = AE$

2. transpose

prove $(AB)^T = B^T A^T$

$(AB)^T_{i,j} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} \cdot B_{ki}$

$(B^T A^T)_{i,j} = \sum_{k=1}^{n} (B_{i,k})^T (A_{k,j})^T = \sum_{k=1}^{n} B_{ki} \cdot A_{jk} = \sum_{k=1}^{n} A_{jk} \cdot B_{ki}$

3. Symmetric matrix 对称矩阵 $A^T = A$

4. matrix inverse 逆矩阵 A^{-1}

$B = A^{-1} \Rightarrow AB = BA = I$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $(ad \neq bc)$

$\Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

① B is unique

Assume $\begin{cases} AB=BA=I \\ AC=CA=I \end{cases}$ $A \neq C$

$B = BI = BAC = IC = C$

② $Ax=b$ $x=A^{-1}b$ (unique)

Assume $\begin{cases} Ax=b \\ Ay=b \end{cases}$ $(x=y) \Rightarrow$ $A(x-y)=\vec{0}$

$A^{-1} \cdot A(x-y) = \vec{0}$

$x=y$

③ diagonal matrix

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{pmatrix}$$

a triangular matrix is invertible iff the diagonal entries are invertible

④ $(AB)^{-1} = B^{-1}A^{-1}$

$(B^{-1}A^{-1}) \cdot AB = B^{-1}(A^{-1}A)B = B^{-1}B = I$ $\Rightarrow$ $(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}$

$AB \cdot (B^{-1} \cdot A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$

[e.g.] let $M=ABC$ where $A,B,C \in \mathbb{R}^{n\times n}$. show M is invertible iff A,B,C are all invertible

① M invertible $\rightarrow$ A,B,C invertible

let $MN = (ABC)N = I$ $\quad$ $A^{-1} = BCN \Rightarrow A$ is invertible

$A^{-1} \cdot A = BCNA = I \Rightarrow$ $\quad$ $B^{-1} = CNA \Rightarrow B$ is invertible

$B^{-1} \cdot B = CNAB = I \Rightarrow$ $\quad$ $C^{-1} = NAB \Rightarrow C$ is invertible

② A,B,C invertible $\rightarrow$ M invertible

$A^{-1}A = BB^{-1} = CC^{-1} = I$

$M(C^{-1}B^{-1}A^{-1}) = I \Rightarrow M^{-1} = C^{-1}B^{-1}A^{-1}$ $\quad$ M is invertible

· 矩阵乘法的几何意义

<1> 左乘 $Aa=b$ $\quad$ b 为 A 列向量组 $\{b_1, b_2 \cdots b_n\}$ 的线性组合

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}$$

<2> 右乘 $a^TA^T = b^T$ $\quad$ b^T 为 A 行向量组 $\{b^{(1)}, b^{(2)}, \cdots b^{(n)}\}$ 的线性组合

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = x\begin{bmatrix} a & b \end{bmatrix} + y\begin{bmatrix} c & d \end{bmatrix}$$

<3> 进一步地 $C = AB$ $C_j = A \cdot B_j$ [类似左乘]

$\Rightarrow C_j$ 是矩阵 A 的列向量的线性组合

$C_{(i)} = A_i \cdot B$ [类似右乘]

$\Rightarrow C_{(i)}$ 是矩阵 B 的行向量的线性组合

$\Rightarrow C = \sum_{i=1}^{n} A_i \cdot B_{(i)}$

$$\overset{3\cdot 2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}} \cdot \overset{2\cdot 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \end{bmatrix}$$

[e.g.1] 见 rank ⑤ 证明

[e.g.2] let $v_1, v_2 \cdots v_d$ be a basis of a subspace $V \in \mathbb{R}^m$

show for any list of vectors $a_1, \cdots a_n$ in V. $\exists X \in \mathbb{R}^{d \times m}$

s.t. $[a_1, \cdots, a_n] = [v_1, \cdots, v_d] X$

$$a_i = [v_1, \cdots v_d] \cdot x_i$$

$$= x_{i1} v_1 + x_{i2} v_2 + \cdots x_{id} v_d$$

is a linear combination $a_i \in \text{span}\{v_1, v_2 \cdots v_d\}$

- GJ-E 的过程

① Forward Elimination (get an upper triangular matrix) REF

② Backward substitution (get a diagonal matrix) RREF $\Rightarrow \begin{bmatrix} I_k & F \\ 0 & 0 \end{bmatrix}$ 非主元的 自由变量

- elementary row operations 初等行变换

① multiplication $A_i \leftarrow CA_i$ ② addition $A_i \leftarrow A_i + \beta A_j$ ③ interchange $\begin{pmatrix} A_i \\ A_j \end{pmatrix} \leftarrow \begin{pmatrix} A_j \\ A_i \end{pmatrix}$

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \alpha A_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix} A_3 + \beta A_1$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_1 \Leftrightarrow A_2$

elementary matrix $\Rightarrow$ a single elementary row operation

the inverse of elementary matrix $\Rightarrow$ reverse operation

$$\begin{cases} E_{R_iR_j}^{-1} = E_{R_iR_j} \\ E_{\alpha R_i}^{-1} = E_{\frac{1}{\alpha}R_i} \\ E_{\beta R_i + R_j}^{-1} = E_{-\beta R_i + R_j} \end{cases}$$

A is invertible $\Leftrightarrow$ n pivots exist in Gauss-Jordan elimination

- if $MA = I_n$. M is invertible, Show $A^{-1} = M$.

$M^{-1} \cdot MA = M^{-1} \Rightarrow A = M^{-1}$ $AM = M^{-1} \cdot M = I$

$\begin{cases} MA = I_n \\ AM = I_n \end{cases} \Rightarrow A$ is invertible and $A^{-1} = M$

- A is invertible iff. $Ax = 0$ has a unique solution $x = 0$

① $\Leftarrow$ "if" G.J-E $(A|0) \Rightarrow$ augumented matrix $(I|0)$

$\exists$ elementary matrices $E_1 \cdots E_p A = I \Rightarrow A$ is invertible

② $\Rightarrow$ "only if"

Assume A is invertible $Ax = 0 \Rightarrow A^{-1}Ax = Ix = 0 \Rightarrow x = 0$

- calculate A^{-1}

① $A \xrightarrow{E_1 \cdots E_k} U \xrightarrow{E_{k+1} \cdots E_p} I$ $A^{-1} = E_p \cdots E_2 E_1$

② $[A | I] \Rightarrow [I | A^{-1}]$

[e.g.] $A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$

$$[A | I] = \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \cdot\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right]$$

• A is invertible [equivalent statements]

- $Ax=0$ has a unique solution $x=0$
- A is a product of elementary matrices
- A has n pivots
- the columns of A
  - span $\mathbb{R}^n$
  - are linearly independent
  - form a basis
- $\dim(C(A))=n$
- $Ax=b$ is solvable for any b

• linear space

1. two operations addition & scalar multiplication

2. 8 axioms:

① $u+v=v+u$ ② $u+(v+w)=(u+v)+w$

③ element 0 $\Rightarrow u+0=u$ ④ $-u$ exists $\Rightarrow u+(-u)=0$

⑤ $\alpha(u+v)=\alpha u+\alpha v$ ⑥ $(\alpha+\beta)u=\alpha u+\beta u$

⑦ $\alpha(\beta u)=(\alpha\beta)u$ ⑧ exists 1 $1u=u$

3. subspace W is a subspace of V ← a linear space if:

① W is a subset of V ② W is a linear space

need to contain W

• V is a subset of V

• $\{0\}$ is a subspace of $\mathbb{R}^n$

① $0+0=0 \Rightarrow$ in $\{0\}$

② $c\cdot 0=0$ $(\forall c\in\mathbb{R})$ in $\{0\}$

③ have 0 element

4. $Ax=b$ has a solution iff $b\in C(A)$. $r(A)=r([A|b])$

let $A=(a_1,\cdots,a_n)$

① $Ax=b$ has a solution $x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$ $a_1x_1+\cdots a_nx_n=Ax=b\in \text{Span}\{a_1\cdots a_n\}=C(A)$

② $b\in C(A)=\text{span}\{a_1,\cdots,a_n\}$ $\exists\ \alpha_1,\cdots,\alpha_n\in\mathbb{R}$ $b=\alpha_1a_1+\cdots+\alpha_na_n$

$\Rightarrow \alpha$ is a solution to $Ax=b$

• $Ax=y$ 值域 $\Rightarrow C(A)$

$[y = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n]$

$x^T A = y^T$ 值域 $R(A)$

$[y^T = x_1 a^{(1)} + x_2 a^{(2)} + \cdots + x_m a^{(m)}]$

每个y至多有个x对应　　每个y至少有个x对应　　个y对应个x

--对应才可能有逆矩阵存在

函数	单射	满射	双射
$Ax=y$	列满秩	行满秩	满秩
$x^TA=y^T$	行满秩	列满秩	

☆判断解的个数 $Ax=b$

$A\in\mathbb{R}^{n\times m}$　　$b\in\mathbb{R}^{n\times 1}$

①是否有解

$\text{rank}(A)=\text{rank}(A|b) \Rightarrow b\in C(A)$　　行满秩 → 一定有解

proof: full row rank $\Leftrightarrow$ $Ax=y$ 满射　　值域为 $Col(A)$

$\Rightarrow b\in C(A)$

② 解的个数

$A\in\mathbb{R}^{m\times n}$　　1° $\text{rank}(A)=\text{rank}(A|b)=n$　　唯一解

列满秩 $\Rightarrow$ 单射

2° $\text{rank}(A)=\text{rank}(A|b)<n$　　非单射　　无数解

[e.g.] $A\in\mathbb{R}^{m\times n}$ of rank r, how many solutions could have in $Ax=b$?

① $m=n=r=6$　　one solution　　$C(A)$ fill $\mathbb{R}^6$　　$b\in\mathbb{R}^6 \Rightarrow b\in C(A)$

② $m=9,\ n=r=7$

<1> $b\notin C(A)$　　0 solution　　<2> $b\in C(A)$　　no free variables　　1 solution

③ $m=r=7,\ n=8$

$C(A)$ fill $\mathbb{R}^7$　　$b\in\mathbb{R}^7 \Rightarrow b\in C(A)$　　$r<n$　　∞ solutions

④ $m=n=8,\ r=7$

<1> $b\notin C(A)$　　0 solution

<2> $b\in C(A)$　　∞ solutions

③ 解集 solution set

$Ax=b \quad x = p + N(A)$ — 一组特解

<1> $A(p+N(A)) = A(p) + A(N(A)) = b + 0 = b$

<2> assume p' satisfied $Ap' = b$

$A(p'-p) = 0 \quad p'-p \in N(A) \Rightarrow p' \in p + N(A)$

$\Rightarrow$ $Ax=b$'s solution set : $x = p + N(A)$

• rank

① $0 \le \text{rank}(A_{m\times n}) \le \min\{n, m\}$

② $\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B)$

rough proof: $\text{rank}(\langle a_1+b_1, \cdots a_n+b_n\rangle) \le \text{rank}(a_1, \cdots a_n, b_1 \cdots b_n) = \text{rank}(A) + \text{rank}(B)$

③ $\text{rank}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} A' & 0 \\ 0 & B' \end{bmatrix}\right) = \text{rank}(A) + \text{rank}(B)$ (A', B' RREF)

elementary row operations don't change rank

if P, Q has a full rank $\Rightarrow \text{rank}(PA) = \text{rank}(AQ) = \text{rank}(PAQ) = \text{rank}(A)$

④ $\text{rank}(A) = \text{rank}(A^T)$ $\quad R(A) = C(A) = \text{rank}(A)$

⑤ $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$ 线性变换

let $C = AB$

$C_{i,j} = \sum_{k=1}^{m} a_{i,k} \cdot b_{k,j}$ then we can see it as linear combination

<1> $C_i = A \cdot B_i \quad (i = 1, 2, \cdots p)$

$= (A_1, A_2 \cdots A_m) \cdot \begin{bmatrix} b_{1,i} \\ b_{2,i} \\ \vdots \\ b_{m,i} \end{bmatrix} = b_{1,i} \cdot A_1 + b_{2,i} \cdot A_2 + \cdots + b_{m,i} \cdot A_m$

$\Rightarrow \text{rank}(C) = \text{rank}(AB) \le C(A) = \text{rank}(A)$

<2> $C_{(i)} = A_{(i)} \cdot B \quad (i = 1, 2, \cdots n)$

$= [a_{i,1}, a_{i,2}, \cdots a_{i,m}] \cdot \begin{bmatrix} B_{(1)} \\ B_{(2)} \\ \vdots \\ B_{(m)} \end{bmatrix} = a_{i,1} \cdot B_{(1)} + a_{i,2} \cdot B_{(2)} + \cdots + a_{i,m} \cdot B_{(m)}$

$\Rightarrow \text{rank}(C) = \text{rank}(AB) \le R(B) = \text{rank}(B)$

[变式] $A \in \mathbb{R}^{m \times n}$ $B \in \mathbb{R}^{n \times k}$

$$\begin{bmatrix} I_n & 0 \\ -A & I_m \end{bmatrix} \begin{bmatrix} I_n & -B \\ A & 0 \end{bmatrix} \begin{bmatrix} I_n & B \\ 0 & I_p \end{bmatrix}$$

- $\operatorname{rank}(AB) \geq \operatorname{rank}(A) + \operatorname{rank}(B) - n$

<1> $n + \operatorname{rank}(AB) = \operatorname{rank}\left(\begin{bmatrix} I_n & 0 \\ 0 & AB \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} I_n & -B \\ A & 0 \end{bmatrix}\right) \geq \operatorname{rank}(A) + \operatorname{rank}(B)$

<2> another proof using rank-nullity theorem.

$A \in \mathbb{R}^{n \times p}$, $B \in \mathbb{R}^{p \times q}$ $AB \in \mathbb{R}^{n \times q}$

if $x \in N(B)$, $Bx = 0 \Rightarrow ABx = 0 \Rightarrow x \in N(AB)$

$\Rightarrow N(B) \subseteq N(AB)$ $\dim(N(B)) \leq \dim(N(AB))$

if $x \notin N(B)$. $x \in N(A) \Rightarrow x \in N(AB)$

$N(AB) \subseteq N(B) \cup B^{-1}[N(A)]$ $\Rightarrow \dim(N(AB)) \leq \dim(N(A)) + \dim(N(B))$

$\Rightarrow \dim[B^{-1}(N(A))] \leq \dim(A)$

$\operatorname{rank}(AB) = q - \dim(N(AB))$, $\operatorname{rank}(A) = p - \dim(N(A))$

$\operatorname{rank}(B) = q - \dim(N(B))$

$\Rightarrow q - \operatorname{rank}(AB) \leq p + q - (\operatorname{rank}(A) + \operatorname{rank}(B))$

$\Rightarrow \operatorname{rank}(AB) \geq \operatorname{rank}(A) + \operatorname{rank}(B) - p$

- $\operatorname{rank}(ABC) = \operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank}(B)$

$\operatorname{rank}(B) + \operatorname{rank}(ABC) = \operatorname{rank}\left(\begin{bmatrix} B & 0 \\ 0 & ABC \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} BC & B \\ 0 & AB \end{bmatrix}\right)$.

$\geq \operatorname{rank}(AB) + \operatorname{rank}(BC)$

6 $\operatorname{rank}(I - AB) \leq \operatorname{rank}(I - A) + \operatorname{rank}(I - B)$

$\operatorname{rank}(I - AB) = \operatorname{rank}(I - A + A - AB) \leq \operatorname{rank}(I - A) + \operatorname{rank}(A - AB)$

$\operatorname{rank}(A - AB) = \operatorname{rank}(A(I - B)) = \min\{\operatorname{rank}(A), \operatorname{rank}(I - B)\}$

$\leq \operatorname{rank}(I - B)$

⑦ $r(A)+r(C) \le r\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \le r(A)+r(B)+r(C)$

let $A \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{m\times n}$

the left q column $r\left(\begin{bmatrix} A \\ 0 \end{bmatrix}\right) \ge r(A)$

the right n column $r\left(\begin{bmatrix} B \\ C \end{bmatrix}\right) \ge r(C)$

$\Rightarrow r(A)+r(C) \le r\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$

$$\underset{M}{\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}} = \underset{X}{\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}} + \underset{Y}{\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}} \Rightarrow \begin{cases} r(X) = r(A)+r(C) \\ r(Y) = r(B) \end{cases}$$

$r(X+Y) \le r(X)+r(Y) \Longrightarrow r\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \le r(A)+r(B)+r(C)$

⑧ every elementary matrix has a full rank

<1> $I_n = \{e_1, e_2, \cdots e_n\}$ has a full rank

<2> let $R_1 \leftarrow R_1 + kR_2$

$\alpha_1(e_1+ke_2) + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$

let $\alpha_1 \ne 0$ $\quad e_1 = -\dfrac{(\alpha_2+k\alpha_1)e_2+\alpha_3 e_3+\cdots+\alpha_n e_n}{\alpha_1}$

which contradicts with $e_1, e_2, \cdots e_n$ is independent

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

<3> $R_1 \leftrightarrow R_2$, $R_1 \leftarrow cR_1$ are same

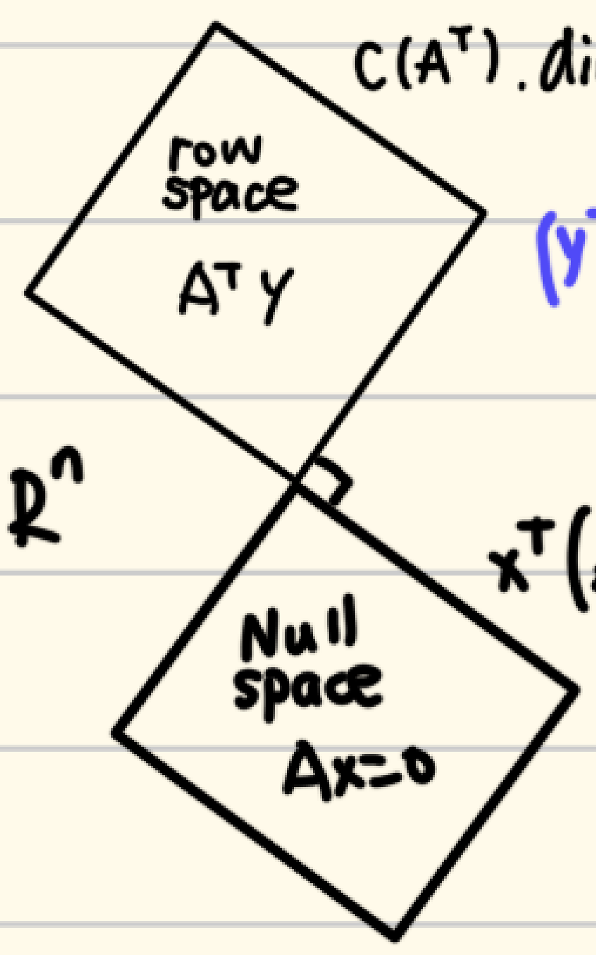
⑨ Fundamental Theorem

(1) $N(A) = C(A)^\perp$ $\quad \dim(N(A)) = n - \mathrm{rank}(A)$ $\qquad Ax = \sum A_i \cdot x_i$

(2) $N(A^T) = R(A)^\perp$ $\quad \dim(N(A^T)) = m - \mathrm{rank}(A)$

$C(A^T)$. dim r $\quad [\ y^T\ ][\ A\ ] = [\quad]$

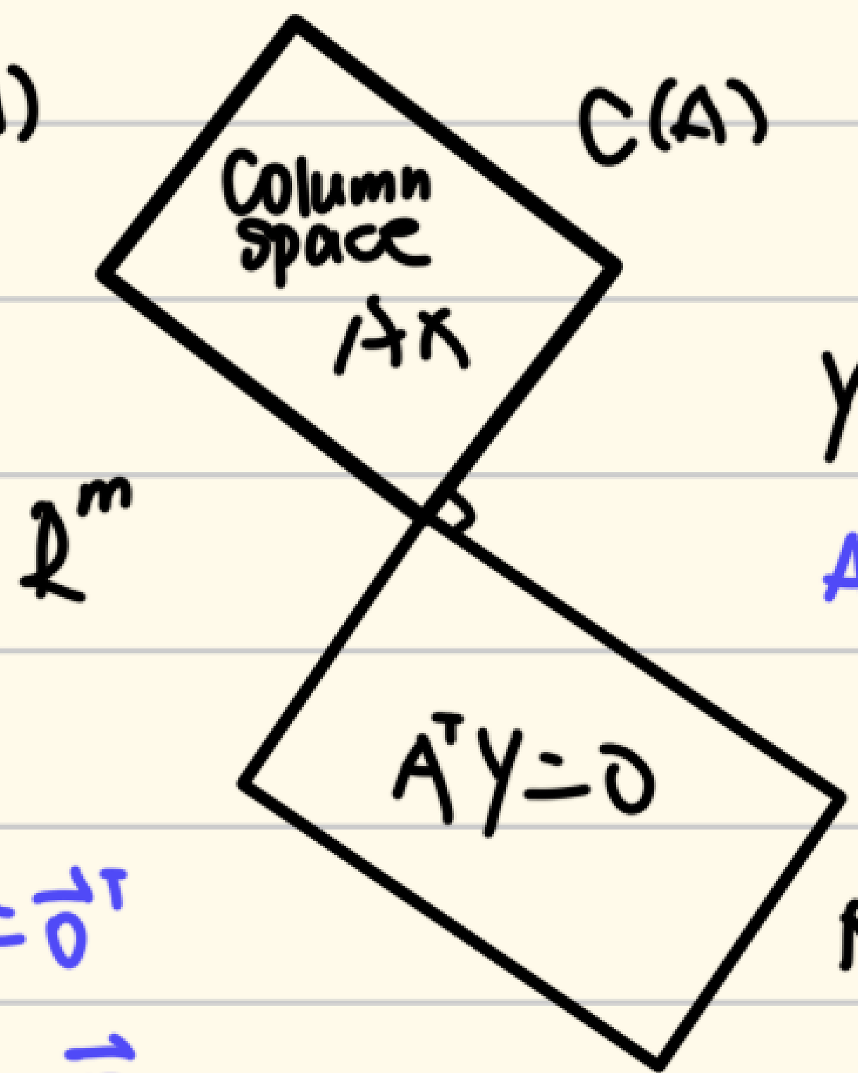
$(y^T A)^T = A^T y$



$x^T(A^T y) = 0$ $\quad$ dim(n-r)

$A\vec{x} = 0 \Rightarrow A_{(i)}\vec{x} = 0$

$\Rightarrow \vec{x} = R(A)^\perp = N(A)$



C(A)

$y^T(Ax) = 0$

$A^T \cdot \vec{y} = 0 \Rightarrow A_i \cdot \vec{y} = 0$

$\Rightarrow \vec{y} = C(A)^\perp = N(A^T)$

$N(A^T)$

$(y^T A) = \vec{0}^T$

$\Rightarrow A^T y = \vec{0}$

[e.g.1] $A \in \mathbb{R}^{m\times n}$. if $B \in \mathbb{R}^{r\times m}$. $BA=C$, $rank(B)=m$, show $rank(A)=rank(C)$

proof: <1> $BAx=Cx=0 \Rightarrow N(A) \subseteq N(C)$

<2> if $x \in N(C)$ $\quad BAx=Cx=0 \qquad rank(B)=m \rightarrow$ injective (單射)

$BAx=0 \Rightarrow Ax=0 \qquad N(C) \subseteq N(A)$

$\Rightarrow N(A)=N(C) \qquad n-rank(A)=n-rank(C) \Rightarrow rank(A)=rank(C)$

[e.g.2] $A, B \in \mathbb{R}^{m\times n}$, prove:

$rank(AB+A+B) \leq rank(A)+rank(B)$

proof: $$\begin{pmatrix} A & A+B \\ 0 & B \end{pmatrix}\begin{pmatrix} B & 0 \\ E & 0 \end{pmatrix} = \begin{pmatrix} AB+A+B & 0 \\ B & 0 \end{pmatrix}$$

$$rank(AB+A+B) \leq rank\begin{pmatrix} AB+A+B & 0 \\ B & 0 \end{pmatrix} \leq \begin{pmatrix} A & A+B \\ 0 & B \end{pmatrix} = rank(A)+rank(B)$$

[e.g.3] $A, B \in \mathbb{R}^{n\times n}$ if $AB=0$, $r(A^2)=r(A)$, prove $\dim(C(A)\cap C(B))=0$

$AB=0 \Rightarrow C(B) \subseteq N(A)$

Assume $x \in C(A)\cap N(A)$, $x \neq 0$ ①

$$\begin{cases} x \in C(A) \Rightarrow x = Ay \\ x \in N(A) \Rightarrow Ax=0 \end{cases} \Rightarrow A^2y=0 \qquad rank(A^2)=rank(A) \Rightarrow N(A^2)=N(A)$$

$\Rightarrow y \in N(A^2) \Rightarrow x=Ay=0$ which contradicts with ①

$\Rightarrow C(A)\cap C(B) \subseteq C(A)\cap N(A)=0$

• orthogonal

投影系數向量

投影向量: $Ay = A\cdot(A^TA^{-1})A^Tb$

1. for a least square problem $Ax=b$, suppose A has linearly independent columns, then the least square solution is $y=(A^TA)^{-1}A^Tb$.

$b-Ay \perp C(A) \Rightarrow (b-Ay)^T\cdot Ax=0 \quad \forall x \in \mathbb{R}^{n\times 1}$

$\Rightarrow (b-Ay)^TA=0 \Rightarrow A^T(b-Ay)=0 \Rightarrow y=(A^TA)^{-1}A^Tb$

[e.g.] find the projection of (2,3,2,1) onto the nullspace of

$$A=\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \qquad Ax=0 \Rightarrow \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}x=0 \qquad \text{the basis of } N(A): \left\{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right\}$$

$$V=\begin{bmatrix} -3 & 0 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \qquad V^TV=\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \qquad \text{projection}: V\cdot(V^TV)^{-1}V^Tb$$

$$= \left[\frac{9}{10}, -\frac{3}{10}, \frac{6}{5}, -\frac{3}{5}\right]^T$$

[e.g.] 3. **Orthogonality** (10 points)

Suppose S is a subspace of $\mathbb{R}^n$. Suppose $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{p} \in S$ satisfy $(\mathbf{b}-\mathbf{p}) \perp S$. Prove that for any $\mathbf{z} \in S$, we have $\|\mathbf{b}-\mathbf{z}\|^2 - \|\mathbf{b}-\mathbf{p}\|^2 = \|\mathbf{z}-\mathbf{p}\|^2$.

Note that if $x \perp y$, then $\|x\|^2 + \|y\|^2 = \|x+y\|^2$

$\|b-z\|^2 = \|b-p+p-z\|^2$. $(b-p) \perp S$. $p-z \in S \Rightarrow b-p \perp p-z$

$\Rightarrow \|b-z\|^2 = \|b-p\|^2 + \|z-p\|^2$

2. $W \subset \mathbb{R}^n$. $\mathcal{B}$ is an orthonormal basis $\{w_1, w_2, \dots, w_k\}$ of W. and a vector $v \in \mathbb{R}^n$. the projection p of v on W is:

$$p = \sum_{i=1}^{n} \langle w_i, v \rangle w_i$$

$\langle v-p, W \rangle = 0 \Rightarrow (v-p) \cdot w_i = v \cdot w_i - w_i \sum_{j=1}^{n} \alpha_j w_j$

$p = \alpha_i w_i$

$= v w_i - w_i \cdot \alpha_i w_i = 0$

$\Rightarrow \alpha_i = v \cdot w_i$

$$\Rightarrow p = \sum_{i=1}^{n} \langle w_i, v \rangle w_i$$

3. orthogonal matrix

<1> $y = Px$ P is orthogonal matrix $\Rightarrow \|x\| = \|y\|$ 只作旋转/翻转

$\|y\| = \sqrt{y^T y} = \sqrt{x^T P^T P x} = \sqrt{x^T P^{-1} P x} = \sqrt{x^T x} = \|x\|$

<2> the eigenvalue of P is 1 or -1

$Av = \lambda v \ (v \neq 0)$ $\quad v^T A^T = \lambda v^T$

$\Rightarrow v^T A^T A v = \lambda v^T \cdot A v$

$v^T v = \lambda v^T \lambda v = \lambda^2 v^T v \quad (v^T v \neq 0)$

$\Rightarrow \lambda = \pm 1$

· Linear transform

1. $f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$

2. find the matrix representation of the linear transformation

[e.g.1] $P_3 \to P_2$ $T: u \to \frac{du}{dx}$

P_3 basis $\{1, x, x^2, x^3\}$

P_2 basis $\{1, x, x^2\}$

$$T(1)=\frac{d}{dx}(1)=0\cdot 1+0\cdot x+0\cdot x^2$$
$$T(x)=\frac{d}{dx}(x)=1\cdot 1+0\cdot x+0\cdot x^2$$
$$T(x^2)=\frac{d}{dx}(x^2)=0\cdot 1+2\cdot x+0\cdot x^2$$
$$T(x^3)=\frac{d}{dx}(x^3)=0\cdot 1+0\cdot x+3\cdot x^2$$

$$\Rightarrow A=\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}$$

[e.g.2] $T: P_2 \to P_3$ $T(f): x \to \int_0^x f(t)\,dt$

P_2 basis $\{1, x, x^2\}$

P_3 basis $\{1, x, x^2, x^3\}$

$$T(1)=\int_0^x 1\,dt=0\cdot 1+1\cdot x+0\cdot x^2+0\cdot x^3$$
$$T(x)=\int_0^x t\,dt=0\cdot 1+0\cdot x+\tfrac{1}{2}\cdot x^2+0\cdot x^3$$
$$T(x^2)=\int_0^x t^2\,dt=0\cdot 1+0\cdot x+0\cdot x^2+\tfrac{1}{3}\cdot x^3$$

$$\Rightarrow A=\begin{bmatrix}0&0&0\\1&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{3}\end{bmatrix}$$

3. basis tranformation

basis α . β $\alpha \to \beta$: $\beta=\alpha\cdot A$ 过渡矩阵

向量n在α.β下坐标为 a, b

则 $\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}=A\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}$

$$(\alpha_1,\cdots,\alpha_n)\cdot\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}=(\beta_1,\cdots,\beta_n)\cdot\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}$$
$$=(\alpha_1,\cdots,\alpha_n)\cdot A\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}$$
$$\Rightarrow\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}=A\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}$$

· characteristic equation

1. $A\lambda = \lambda x \Rightarrow (A-\lambda I)x = 0 \Rightarrow |A-\lambda I| = 0$

* if $\lambda_1, \lambda_2 \cdots \lambda_m$ 是n阶方阵A的 eigenvalues, (不同) 则 eigenvectors $\{v_1 \cdots v_m\}$ is linear independent.

Proof: ① $m=1$, $v_1 \neq 0 \Rightarrow \{v_1\}$ is linear independent

② Assume $m=k-1$ $\{v_1, v_2, \cdots, v_{k-1}\}^*$ is linear independent. then Assume after adding v_k $\{v_1, \cdots, v_k\}$ is linear dependent.

then $c_1 v_1 + \cdots + c_{k-1} v_{k-1} = v_k$ ① $\Rightarrow c_1 A v_1 + \cdots c_{k-1} A v_{k-1} = A v_k$

since $Av = \lambda v$. $c_1 \lambda_1 v_1 + \cdots + c_{k-1}\lambda_{k-1} v_{k-1} = c_k \lambda_k v_k$ ②

① $\times \lambda_k$ - ②: $c_1(\lambda_k - \lambda_1) v_1 + \cdots + c_{k-1}(\lambda_k - \lambda_{k-1}) v_{k-1} = 0$

by *. we can conclude $c_1(\lambda_k - \lambda_1) = \cdots = c_{k-1}(\lambda_k - \lambda_{k-1}) = 0$

$\lambda_1 \cdots \lambda_k$ is pairwise distinct $\Rightarrow c_1 = c_2 = \cdots = c_{k-1} = 0$

by ①. $v_k = 0$, wich contradict with $v_k \neq 0$ #

matrix	A	$aA+bE$	A^n	A^{-1}	A^*	A^T
eigenvalue	λ	$a\lambda+b$	λ^n	$\frac{1}{\lambda}$	$\frac{\|A\|}{\lambda}$	λ
eigenvecter	ξ	ξ	ξ	ξ	ξ	/

$A\xi = \lambda\xi$

$(aA+bE)\xi = aA\xi + bE\xi = a\lambda\xi + b\xi = (a\lambda+b)\xi$

$A^n\xi = A^{n-1}A\xi = \lambda A^{n-1}\xi = \lambda^n\xi$

$A\xi = \lambda\xi \Rightarrow \frac{1}{\lambda}\xi = A^{-1}\xi$

$\frac{1}{\lambda}\xi = A^{-1}\xi \Rightarrow |A| \cdot A^{-1}\xi = \frac{|A|}{\lambda}\xi \Rightarrow A^*\xi = \frac{|A|}{\lambda}\xi$

$|A^T - \lambda E| = |A^T - \lambda E^T| = |(A-\lambda E)^T| = |A - \lambda E|$

[e.g.] $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$, $(\lambda_1 \neq \lambda_2)$. Show $k_1v_1 + k_2v_2$ is not eigenvector of A

proof:

let $k_1 = k_2 = 1$, and assume $A(v_1+v_2) = \lambda(v_1+v_2)$

$\Rightarrow Av_1 + Av_2 = \lambda_1 v_1 + \lambda_2 v_2 = \lambda(v_1+v_2)$

$(\lambda_1 - \lambda)v_1 + (\lambda_2 - \lambda)v_2 = 0 \Rightarrow v_1, v_2$ is linear independent

$\Rightarrow \lambda = \lambda_1 = \lambda_2$ which contradicts with $\lambda_1 \neq \lambda_2$

2. diagonalize

$P=(p_1,\cdots,p_n)$ (P is a linear independent eigenvectors)

$$AP=(Ap_1,\cdots,Ap_n)=(\lambda_1 p_1,\lambda_2 p_2,\cdots,\lambda_n p_n)$$
$$=(p_1,\cdots,p_n)\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}=P\cdot\Lambda$$

$\Rightarrow A=P\Lambda \boxed{P^{-1}}$ 涉及求逆，为简便计算 → 正交对角化 ($A^TA=E \quad A^{-1}=A^T$)

3. real symmetric matrices [实对称矩阵]

$A\in\mathbb{R}^{n\times n}$ and $A=A^T$

① all eigenvalues of A are real

$\Rightarrow$ try to prove $\bar{\lambda}=\lambda$ $\qquad$ [$V^*=\bar{V}^T$ 共轭转置]

$$V^*AV=\lambda V^*V=\lambda\|V\|^2$$
$$(V^*AV)^*=V^*A^*V=V^*AV$$
$$A^*=\bar{A}^T=A^T=A$$

$$\Rightarrow V^*AV=\lambda\|V\|^2=(V^*AV)^*=(\lambda\|V\|^2)^*$$
$$=\bar{\lambda}\|V\|^2$$
$$\Rightarrow \lambda=\bar{\lambda}$$

② eigenvectors corresponding to different eigenvalues are orthogonal.

$$\begin{cases}Av_1=\lambda_1 v_1\\ Av_2=\lambda_2 v_2\end{cases}\Rightarrow \quad \begin{aligned}&v_2^*A^*=\lambda_2 v_2^*\\ &v_2^*A^*v_1=\lambda_2 v_2^*v_1\\ &v_2^*Av_1=\lambda_2 v_2^*v_1\\ &v_2^*\lambda_1 v_1=\lambda_2 v_2^*v_1\end{aligned}$$

$\Rightarrow \lambda_1 v_2^*v_1=\lambda_2 v_2^*v_1$

Since $\lambda_1\neq\lambda_2 \Rightarrow v_2^*v_1=0$

i.e. v_1, v_2 are orthogonal

[e.g.] $A=\begin{pmatrix}0&-1&1\\-1&0&1\\1&1&0\end{pmatrix}$ diagonalize

$$\begin{vmatrix}-\lambda&-1&1\\-1&-\lambda&1\\1&1&-\lambda\end{vmatrix}=0\Rightarrow\lambda_1=-2,\ \lambda_2=\lambda_3=1$$

$\Rightarrow p_1=\begin{pmatrix}-1\\-1\\1\end{pmatrix},\ p_2=\begin{pmatrix}-1\\1\\0\end{pmatrix},\ p_3=\begin{pmatrix}1\\0\\1\end{pmatrix}$

$v_1=\frac{1}{\sqrt3}\begin{pmatrix}-1\\-1\\1\end{pmatrix},\ v_2=\frac{1}{\sqrt2}\begin{pmatrix}-1\\1\\0\end{pmatrix},\ v_3=\frac{1}{\sqrt6}\begin{pmatrix}1\\1\\2\end{pmatrix}$ $\Rightarrow P=\begin{pmatrix}-\frac{1}{\sqrt3}&-\frac{1}{\sqrt2}&\frac{1}{\sqrt6}\\-\frac{1}{\sqrt3}&\frac{1}{\sqrt2}&\frac{1}{\sqrt6}\\\frac{1}{\sqrt3}&0&\frac{2}{\sqrt6}\end{pmatrix}$

③ spectral theorem 谱定理 [不是所有矩阵能被对角化 几何重数=代数重数]

Any real symmetric matrix A can be written as $A=VDV^T$

(V is a real orthogonal matrix / D is a real diagonal matrix)

4. similar matrices

$A, B\in\mathbb{R}^{n\times n}$ satisfy $A=SBS^{-1}$ for some $S\in\mathbb{R}^{n\times n}$, invertible

$\Rightarrow A$ and B are similar ($A\sim B$)

① $\mathrm{EIG}(A)=\mathrm{EIG}(B)$

$P_A(\lambda)=\det(A-\lambda I)=\det(SBS^{-1}-\lambda SS^{-1})=\det(S(B-\lambda)S^{-1})$

$=\det(B-\lambda)\cdot|S|\cdot\frac{1}{|S|}=\det(B-\lambda I)=P_B(\lambda)$

② $tr(A)=tr(B)$ $\quad\det(A)=\det(B)$

$tr(A)=tr((S^{-1}B)S)=tr(S(S^{-1}B))=tr(B)$ ★ $tr(MN)=tr(NM)$

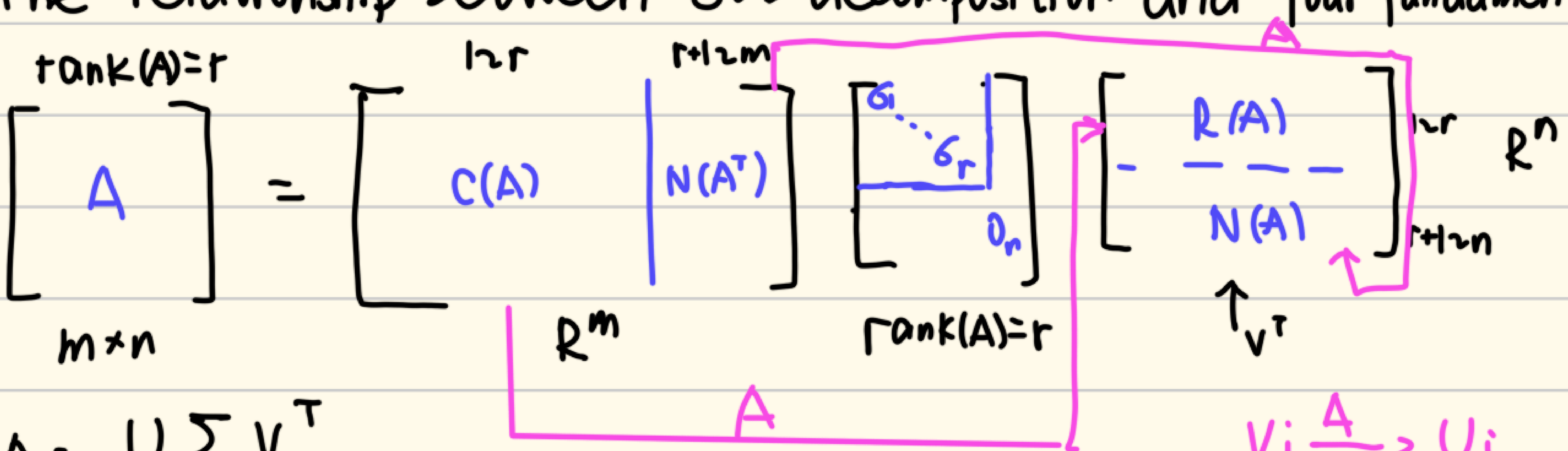
$\det(A)=\det(S^{-1}BS)=\det(S^{-1})\cdot\det(S)\cdot\det(B)=\det(B)$

5. SVD decomposition

A 非对称 $\Rightarrow AA^T/A^TA$ 实对称

$$\begin{cases}AA^T=PS^2P^T\\ A^TA=QS^2Q^T\end{cases}\ \begin{matrix}\Leftarrow\\ \not\Rightarrow\end{matrix}\ A=P\underbrace{S}_{\sigma_i=\sqrt{\lambda_i}>0}Q^T\Rightarrow AQ=PS\Rightarrow p_i=\frac{Aq_i}{s_i}$$

· the relationship between SVD decomposition and four fundamental subspaces



$A=U\Sigma V^T$ $\qquad v_i\xrightarrow{A}u_i$

$A\vec{v_i}=\vec{u_i}\cdot\vec{\sigma_i}$

$U_1=[u_1,\cdots u_r]$, $U_2=[u_{r+1}\cdots u_m]$

$V_1=[v_1\cdots v_r]$, $V_2=[v_{r+1}\cdots v_n]$

① $A\in\mathbb{R}^{m\times n}$. $R(A)=\{Ax\mid x\in\mathbb{R}^n\}\subseteq\mathbb{R}^m$

$A=\sum_{i=1}^{n}\sigma_i u_i v_i^T=\sum_{i=1}^{r}\sigma_i u_i v_i^T$

$y=Ax=\sum_{i=1}^{r}(\sigma_i v_i^T x)u_i=\sum_{i=1}^{r}\alpha_i u_i=C(A)=C(U_1)$

② $Ax=\sum_{i=1}^{r}\sigma_i u_i v_i^T x$ $\qquad$ let $z=\sum_{i=r+1}^{n}\beta_i v_i$

$Az=\left(\sum_{i=1}^{r}\sigma_i u_i v_i^T\right)\left(\sum_{i=r+1}^{n}\beta_i v_i\right)=0$ $\qquad\Rightarrow N(A)=\{x\mid Ax=0\}=C(V_2)$

③ $R(A^T)=R(A)=C(V_1)$

④ $N(A^T)=C(U_2)$

· application

1. pagerank [eigenvalue]

① construct a matrix A(其他点链接到该点的概率)

② $V^{(t+1)} = AV^{(t)}$ iteration

③ $V = AV$ 趋于稳定

$\Rightarrow$ V is a eigenvector of A whose eigenvalue is 1

2. Dynamics [diagonalization]

$$\underbrace{\begin{pmatrix} U_{k+1} \\ V_{k+1} \end{pmatrix}}_{\vec{z}_{k+1}} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ B_1 & B_2 \end{pmatrix} \underbrace{\begin{pmatrix} U_k \\ V_k \end{pmatrix}}_{\vec{z}_k} \Rightarrow \vec{z}_k = A^k \cdot \vec{z}_0$$

[e.g.] predator-Prey Model

Q12 Given a year $k \in \mathbb{N}$, let us denote $x_k, y_k \in \mathbb{N}$, respectively, the number of rabbits and the number of wolves and let us assume they satisfy the equations:

$$\begin{cases} x_{k+1} = 1.2x_k - 0.2y_k \\ y_{k+1} = 0.3x_k + 0.5y_k. \end{cases}$$

Introducing $A = \begin{pmatrix} 1.2 & -0.2 \\ 0.3 & 0.5 \end{pmatrix}$, $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\lambda_1 = 1.1$, $\lambda_2 = 0.6$, one can show that $Au = \lambda_1 u$ and $Av = \lambda_2 v$. Assuming that at year 0, $x_0 = 100$, $y_0 = 40$, deduce a simple expression of x_k, y_k for any $k \in \mathbb{N}$ as a function of k, λ_1, λ_2.

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 1.2 & -0.2 \\ 0.3 & 0.5 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} \qquad A = S\Lambda S^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1.1 & 0 \\ 0 & 0.6 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1.1^k & 0 \\ 0 & 0.6^k \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 100 \\ 40 \end{pmatrix}$$

3. compression [SVD] low-rank approximation

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{m} (a_{ij})^2}$$

$$A_k = \sum_{i=1}^{k} \sigma_i U_i V_i^T \qquad \underset{X}{\text{minimize}} \; \|A - X\|_F \qquad \text{rank}(X) = K \qquad X = A_k$$

$(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq \sigma_{k+1} \geq \cdots \geq \sigma_r > 0)$

① $\|B\|_F^2 = tr(B^T B)$

$C = B^T B \qquad C_{i,j} = \sum_{k=1}^{m} B^T_{i,k} \cdot B_{kj} = \sum_{k=1}^{m} B_{ki} \cdot B_{kj} \qquad tr(B^T B) = \sum_{i=1}^{n} \sum_{k=1}^{m} B_{ki}^2 = \|B\|_F^2$

② $\|A\|_F = \sqrt{tr(AA^T)} = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$

$\Rightarrow \|A - A_k\|_F = \sqrt{\sigma_{k+1}^2 + \cdots + \sigma_r^2}$

4. curve classification [quadratic forms]

$x^T A x = C$

① $A = UDU^T$

② $z = U^T x$

③ $x^T A x = z^T D z = C \Rightarrow \sum d_i z_i^2 = C$

④ eigenvalues of A $\begin{cases} ++ / -- & \text{ellipsoid} \\ +- & \text{hyperbola} \\ \lambda_i = 0 & \text{parabola} \end{cases}$

[e.g.] $5x_1^2 - 6x_1x_2 + 5x_2^2 = 4$

$A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$ $\lambda_1 = 8$, $\lambda_2 = 2$ $z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T \cdot x$ $\Rightarrow \frac{z_1^2}{0.5} + \frac{z_2^2}{2} = 1$ ellipsoid